Gradient Discretisations
Tools and Applications

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- Ω is an open bounded connected subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^\star$)
- $p \in (1, +\infty)$
- $r \in L^{p'}(\Omega)$ and $\boldsymbol{R} \in L^{p'}(\Omega)^d$ with $p' = \dfrac{p}{p-1}$.

Strong sense (homogeneous Dirichlet BC) :

Find $\overline{u} \in W_0^{1,p}(\Omega)$ such that $-\mathrm{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u} + \boldsymbol{R}) = r$

Weak sense :

Find $\overline{u} \in W_0^{1,p}(\Omega)$ such that, for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \overline{u}(\boldsymbol{x})|^{p-2}\nabla \overline{u}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega r(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{R}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

Using the Gradient Discretisation, Let us have schematisation

Gradient Discretisation for homogeneous Dirichlet BC :

$X_{\mathcal{D},0}$ vector space of degrees of freedom
$\Pi_{\mathcal{D}} \ : \ X_{\mathcal{D},0} \to L^p(\Omega)$ linear function reconstruction
$\nabla_{\mathcal{D}} \ : \ X_{\mathcal{D},0} \to L^p(\Omega)^d$ linear gradient reconstruction
$\|\nabla_{\mathcal{D}} \cdot \|_{L^p}$ norm on $X_{\mathcal{D},0}$

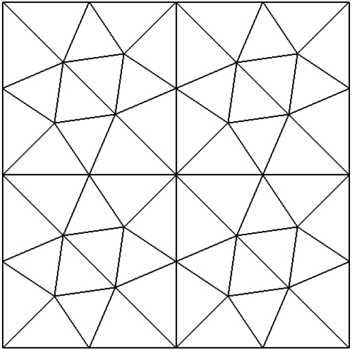
Then scheme :

Find $u \in X_{\mathcal{D},0}$ such that, for any $v \in X_{\mathcal{D},0}$,

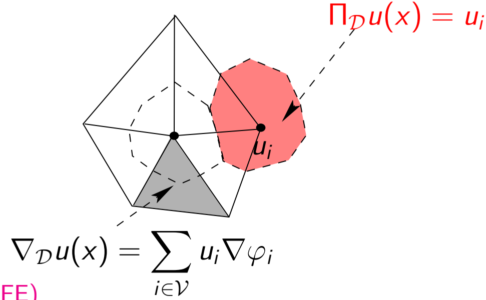
$$\int_\Omega |\nabla_{\mathcal{D}} u(\boldsymbol{x})|^{p-2} \nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega r(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{R}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

Three examples of space approximations
Which can be seen as Gradient Discretisations

On a triangular mesh, $X_{\mathcal{D},0} = \{(u_i)_{i\in\mathcal{V}} \in \mathbb{R}^{\mathcal{V}},\ u_i = 0 \text{ if } \boldsymbol{x}_i \in \partial\Omega\}$

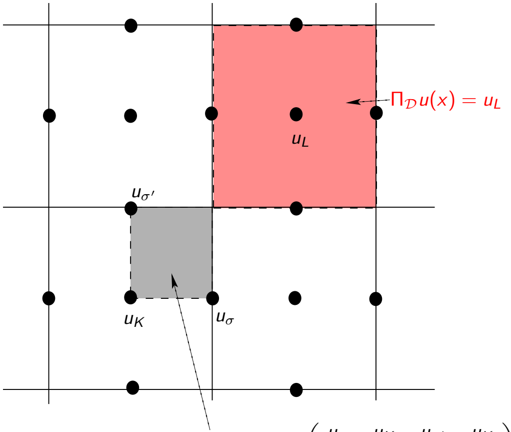


mass lumped $\mathbb{P}_1$ finite elements (CVFE)



The meshes are more regular, When grid blocks are rectangular

On a rectangular mesh, $X_{\mathcal{D},0} = \{(u_K)_{K\in\mathcal{M}}\} \times \{(u_\sigma)_{\sigma\in\mathcal{F}}, u_\sigma = 0 \text{ if } \sigma \subset \partial\Omega\}$



$$\nabla_{\mathcal{D}} u(x) = \left(\frac{u_\sigma - u_K}{\mathrm{d}(\boldsymbol{x}_K, \boldsymbol{x}_\sigma)}, \frac{u_{\sigma'} - u_K}{\mathrm{d}(\boldsymbol{x}_K, \boldsymbol{x}_{\sigma'})} \right)^t$$

Leads to 5-point finite difference scheme if $p = 2$

cannot be seen as non-conforming finite elements ($\nabla_{\mathcal{D}} u$ cannot be deduced from $\Pi_{\mathcal{D}} u$)

For discontinuous Galerkin,
Put the stabilization therein.

Example of the Average Discontinuous Galerkin Gradient Discretisation (ADGGD)

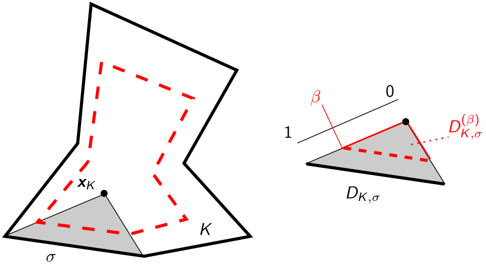
$V_h = \{\ v \in L^2(\Omega)$ such that, for all $K \in \mathcal{M}, v_{|K} \in \mathbb{P}^1(\mathbb{R}^d)\ \}$, $(\psi_i)_{i\in I}$ a basis of V_h

- $X_{\mathcal{D},0} \ = \ \{\ (u_i)_{i\in I}\ \}$ and $\Pi_{\mathcal{D}} u \ = \ \sum_{i\in I} u_i\psi_i \in V_h$
- Let $\beta \in]0,1[$ given. For $u \in X_{\mathcal{D},0}$, for $K \in \mathcal{M}$ and for any $\sigma \in \mathcal{F}_\sigma$:

$$\nabla_{\mathcal{D}} u = \begin{cases} \nabla(\Pi_{\mathcal{D}} u_{|K}) & \text{in } D_{K,\sigma}^{(\beta)} \\ \nabla(\Pi_{\mathcal{D}} u_{|K}) + \dfrac{d}{(1-\beta)} \dfrac{[u]^a_{K,\sigma}}{\mathrm{d}(\boldsymbol{x}_K,\sigma)} \boldsymbol{n}_{K,\sigma} & \text{in } D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)} \end{cases}$$

if $\mathcal{M}_\sigma = \{K, L\}$, $[u]^a_{K,\sigma} = (u^a_{L,\sigma} - u^a_{K,\sigma})/2$
if $\mathcal{M}_\sigma = \{K\}$, $[u]^a_{K,\sigma} = 0 - u^a_{K,\sigma}$
with $\quad u^a_{K,\sigma} = \dfrac{1}{|\sigma|}\displaystyle\int_\sigma \Pi_{\mathcal{D}} u_{|K}(\boldsymbol{x})\mathrm{d}\gamma(\boldsymbol{x})$

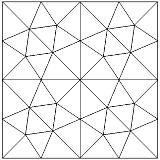
ADGGD is pleasant :
Piecewise constant
Gradient approximations
Provide simple computations



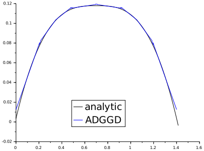
Comparison in two dimensions
With analytical solutions

$d = 2$, $\Omega = (0,1)^2$, $r(\boldsymbol{x}) = 2$, $\boldsymbol{R}(x) = 0$

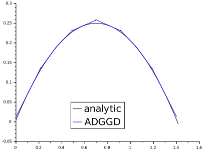
$$\overline{u}(\boldsymbol{x}) = \frac{p-1}{p}\left[\left(\frac{1}{\sqrt{2}}\right)^{p/(p-1)} - |\boldsymbol{x} - \boldsymbol{x}_\Omega|^{p/(p-1)}\right]$$



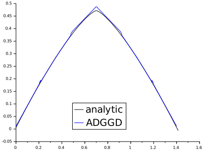
mesh1_1



$p = 1.5$



$p = 2$



$p = 4$

profiles along the diagonal for ADGGD with $\beta = 0.8$

there exists $C_1 > 0$, depending on p, r, $\boldsymbol{R}$ and increasingly depending on $C_{\mathcal{D}}$, such that (recalling that $\boldsymbol{V} := |\nabla \overline{u}|^{p-2}\nabla \overline{u} + \boldsymbol{R} \in W^{p'}_{\mathrm{div}}(\Omega)$ since $\mathrm{div}\, \boldsymbol{V} = -r$) :

If $p \in (1,2]$, $$\frac{1}{C_1}\left(W_{\mathcal{D}}(\boldsymbol{V})^{\frac{1}{p-1}} + S_{\mathcal{D}}(\overline{u})\right) \le \|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \le C_1\left(W_{\mathcal{D}}(\boldsymbol{V}) + S_{\mathcal{D}}(\overline{u})^{p-1}\right)$$

If $p \in [2,+\infty)$, $$\frac{1}{C_1}\left(W_{\mathcal{D}}(\boldsymbol{V}) + S_{\mathcal{D}}(\overline{u})\right) \le \|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \le C_1\Big(W_{\mathcal{D}}(\boldsymbol{V}) + S_{\mathcal{D}}(\overline{u})\Big)^{\frac{1}{p-1}}$$

with

$$C_{\mathcal{D}} = \max_{w \in X_{\mathcal{D}}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} w\|_{L^p(\Omega)}}{\|\nabla_{\mathcal{D}} w\|_{L^p(\Omega)^d}}$$

for all $\varphi \in W^{1,p}_0(\Omega)$, $S_{\mathcal{D}}(\varphi) = \min_{w \in X_{\mathcal{D}}} \left(\|\Pi_{\mathcal{D}} w - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} w - \nabla\varphi\|_{L^p(\Omega)^d}\right)$

for all $\boldsymbol{\varphi} \in W^{p'}_{\mathrm{div}}(\Omega) := \{\boldsymbol{\varphi} \in L^{p'}(\Omega)^d, \mathrm{div}\boldsymbol{\varphi} \in L^{p'}(\Omega)\}$,

$$W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{w \in X_{\mathcal{D}}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} w\|_{L^p(\Omega)^d}} \left|\int_\Omega (\nabla_{\mathcal{D}} w(x)\cdot \boldsymbol{\varphi}(x) + \Pi_{\mathcal{D}} w(x)\,\mathrm{div}\,\boldsymbol{\varphi}(x)\,\mathrm{d}x)\right|$$

shows that $W_{\mathcal{D}}(\boldsymbol{V}) \to 0$ and $S_{\mathcal{D}}(\overline{u}) \to 0$ mandatory. Optimal only if $p = 2$.

For the 3 previous examples : if $p = 2$ and $d \le 3$, $\overline{u} \in W^{2,p}(\Omega)$ and $|\nabla \overline{u}|^{p-2}\nabla \overline{u} + \boldsymbol{R} \in W^{1,p'}(\Omega)^d$, order $h_{\mathcal{D}}^{p-1}$ if $p \in (1,2]$, and $h_{\mathcal{D}}^{\frac{1}{p-1}}$ if $p \ge 2$

To the three examples, applies a series Of GDM core properties

For a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of such GDs with $h_m \to 0$ under a regularity property,

$(\mathcal{D}_m)_{m\in\mathbb{N}}$ is **coercive** : $C_{\mathcal{D}_m} \le C_P$

$(\mathcal{D}_m)_{m\in\mathbb{N}}$ is **consistent** : for all $\varphi \in W_0^{1,p}(\Omega)$, $S_{\mathcal{D}_m}(\varphi) \to 0$

$(\mathcal{D}_m)_{m\in\mathbb{N}}$ is **limit-conforming** : for all $\boldsymbol{\varphi} \in W_{\rm div}^{p'}(\Omega)$, $W_{\mathcal{D}_m}(\boldsymbol{\varphi}) \to 0$

$(\mathcal{D}_m)_{m\in\mathbb{N}}$ is **compact** :
for all $(u_m)_{m\in\mathbb{N}}$, with $u_m \in X_{\mathcal{D}_m}$ such that $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^p(\Omega)^d})_{m\in\mathbb{N}}$ bounded,
exists subsequence of $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converging in $L^p(\Omega)$

Example of non-homogeneous Neumann BC

let $r \in L^{p'}(\Omega)$, $\boldsymbol{R} \in L^{p'}(\Omega)^d$ and $g \in L^{p'}(\partial\Omega)$ s.t. $\displaystyle\int_\Omega r \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g \mathrm{d}s = 0$

find $\overline{u} \in W^{1,p}(\Omega)$ with $\displaystyle\int_\Omega \overline{u} \mathrm{d}\boldsymbol{x} = 0$ and $-\mathrm{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u} + \mathrm{div}\boldsymbol{R}) = r$

with non-hom. Neumann BC $(|\nabla\overline{u}|^{p-2}\nabla\overline{u} + \boldsymbol{R}) \cdot \boldsymbol{n} = g$ on $\partial\Omega$

weak solution $\overline{u}$

$\overline{u} \in W^{1,p}(\Omega)$ and, for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u} \cdot \nabla v \mathrm{d}\boldsymbol{x} + \Big|\int_\Omega \overline{u} \mathrm{d}\boldsymbol{x}\Big|^{p-2} \int_\Omega \overline{u} \mathrm{d}\boldsymbol{x} \int_\Omega v \mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega r v \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{R} \cdot \nabla v \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g \gamma v \mathrm{d}s$$

Scheme with GD : $\mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, \mathbb{T}_\mathcal{D})$

$\overline{u} \in X_\mathcal{D}$ and, for all $v \in X_\mathcal{D}$,

$$\int_\Omega |\nabla_\mathcal{D} u|^{p-2}\nabla_\mathcal{D} u \cdot \nabla_\mathcal{D} v \mathrm{d}\boldsymbol{x} + \Big|\int_\Omega \Pi_\mathcal{D} u \mathrm{d}\boldsymbol{x}\Big|^{p-2} \int_\Omega \Pi_\mathcal{D} u \mathrm{d}\boldsymbol{x} \int_\Omega \Pi_\mathcal{D} v \mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega r \Pi_\mathcal{D} v \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{R} \cdot \nabla_\mathcal{D} v \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g \mathbb{T}_\mathcal{D} v \mathrm{d}s$$

How should we define coercivity, consistency,
Limit-conformity, compactness for all types of BC ?

Continuous functional setting

1. L and $\boldsymbol{L}$ separable reflexive Banach spaces
2. V closed subspace of L' ($V = \{0\}$ possible)
3. $W_{\mathrm{G}} \subset L$ dense subspace and $\mathrm{G} : W_{\mathrm{G}} \to \boldsymbol{L}$ linear operator with closed graph
4. $$\|u\|_{W_{\mathrm{G}}} = \sup_{\mu \in V\setminus\{0\}} \frac{|\langle \mu, u\rangle_{L',L}|}{\|\mu\|_{L'}} + \|\mathrm{G}u\|_{\boldsymbol{L}}$$
5. $\boldsymbol{W}_{\mathrm{D}} = \{\boldsymbol{v} \in \boldsymbol{L}' : \exists w \in L', \forall u \in W_{\mathrm{G}}, \langle \boldsymbol{v}, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle w, u\rangle_{L',L} = 0\}$
denote $\mathrm{D}\boldsymbol{v} := w$
6. $$\forall u \in W_{\mathrm{G}},\ \forall \boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}},\ \langle \boldsymbol{v}, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L} = 0$$

Remark : $\|u\|_L \leq C\,\|u\|_{W_{\mathrm{G}}} \Leftrightarrow \mathrm{Im}(\mathrm{D}) + V = L'$

Discrete functional setting $\mathcal{D} = (X_{\mathcal{D}}, \mathrm{P}_{\mathcal{D}}, \mathrm{G}_{\mathcal{D}})$

1. $X_{\mathcal{D}}$ finite dimensional vector space on $\mathbb{R}$
2. $\mathrm{P}_{\mathcal{D}} : X_{\mathcal{D}} \to L$ a linear mapping
3. $\mathrm{G}_{\mathcal{D}} : X_{\mathcal{D}} \to \boldsymbol{L}$ linear mapping
4. $$\|u\|_{\mathcal{D}} := \sup_{\mu \in V\setminus\{0\}} \frac{|\langle \mu, \mathrm{P}_{\mathcal{D}}u\rangle_{L',L}|}{\|\mu\|_{L'}} + \|\mathrm{G}_{\mathcal{D}}u\|_{\boldsymbol{L}}$$ assumed to be a norm on $X_{\mathcal{D}}$

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D}} \setminus \{0\}} \frac{\|\mathrm{P}_{\mathcal{D}} v\|_L}{\|v\|_{\mathcal{D}}}$$

coercive if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_m} \le C_P$ for all $m \in \mathbb{N}$

$S_{\mathcal{D}} : W_{\mathrm{G}} \to [0, +\infty)$ be given by

$$\forall \varphi \in W_{\mathrm{G}}\,, \quad S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D}}} \left(\|\mathrm{P}_{\mathcal{D}} v - \varphi\|_L + \|\mathrm{G}_{\mathcal{D}} v - \mathrm{G}\varphi\|_{\boldsymbol{L}} \right)$$

consistent if $\forall \varphi \in W_{\mathrm{G}}\,, \ \lim_{m \to \infty} S_{\mathcal{D}_m}(\varphi) = 0$

$W_{\mathcal{D}} : \boldsymbol{W}_{\mathrm{D}} \to [0, +\infty)$

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}\,, \ W_{\mathcal{D}}(\boldsymbol{\varphi}) = \sup_{u \in X_{\mathcal{D}} \setminus \{0\}} \frac{|\langle \boldsymbol{\varphi}, \mathrm{G}_{\mathcal{D}} u \rangle_{\boldsymbol{L}', \boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{\varphi}, \mathrm{P}_{\mathcal{D}} u \rangle_{L', L}|}{\|u\|_{\mathcal{D}}}.$$

limit-conforming if $\forall \boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}\,, \ \lim_{m \to \infty} W_{\mathcal{D}_m}(\boldsymbol{\varphi}) = 0$

$(\mathcal{D}_m)_{m \in \mathbb{N}}$ **compact** if, for any sequence $u_m \in X_{\mathcal{D}_m}$ such that $(\|u_m\|_{\mathcal{D}_m})_{m \in \mathbb{N}}$ bounded, $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$ is relatively compact in L

$\boldsymbol{a} : \boldsymbol{L} \to \boldsymbol{L}'$ continuous, monotonous, coercive, bounded in some sense

$a : L \to V$ weakly continuous, monotonous, coercive, bounded in some sense

Strong formulation of the problem

Find $\overline{u} \in W_{\mathrm{G}}$ such that $-\mathrm{D}(\boldsymbol{a}(\mathrm{G}\overline{u}) + \boldsymbol{F}) + a(\overline{u}) = f$

Weak formulation of the problem

Find $\overline{u} \in W_{\mathrm{G}}$ such that, $\forall v \in W_{\mathrm{G}}$,

$$\langle \boldsymbol{a}(\mathrm{G}\overline{u}), \mathrm{G}v \rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\overline{u}), v \rangle_{L',L} = \langle f, v \rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}v \rangle_{\boldsymbol{L}',\boldsymbol{L}}$$

GDM approximation

Find $u \in X_{\mathcal{D}}$ such that, $\forall v \in X_{\mathcal{D}}$,

$$\langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}}u), \mathrm{G}_{\mathcal{D}}v \rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}u), \mathrm{P}_{\mathcal{D}}v \rangle_{L',L} = \langle f, \mathrm{P}_{\mathcal{D}}v \rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v \rangle_{\boldsymbol{L}',\boldsymbol{L}}$$

Convergence theorem under consistency and limit-conformity

	homogeneous Dirichlet	non-homogeneous Neumann
$\boldsymbol{L}$	$L^p(\Omega)^d$	$L^p(\Omega)^d$
L	$L^p(\Omega)$	$L^p(\Omega) \times L^p(\partial\Omega)$
$V \subset L'$	$\{0\}$	$\mathbb{R}(1_\Omega, 0)$
$W_{\mathrm{G}} \subset L$	$W_0^{1,p}(\Omega)$	$\{(u, \gamma u) \ : \ u \in W^{1,p}(\Omega)\}$
$\Vert u \Vert_{W_{\mathrm{G}}}$	$\Vert \nabla u \Vert_{L^p}$	$\Vert \nabla u \Vert_{L^p} + \vert \int_\Omega u \vert$
$\mathrm{G} : W_{\mathrm{G}} \to \boldsymbol{L}$	$u \mapsto \nabla u$	$(u, w) \mapsto \nabla u$
$\boldsymbol{W}_{\mathrm{D}} \subset \boldsymbol{L}'$	$W_{\mathrm{div}}^{p'}(\Omega)$	$W_{\mathrm{div},\partial}^{p'}(\Omega)$
$\mathrm{D} : \boldsymbol{W}_{\mathrm{D}} \to L'$	$\boldsymbol{v} \mapsto \mathrm{div}\,\boldsymbol{v}$	$\boldsymbol{v} \mapsto (\mathrm{div}\,\boldsymbol{v}, -\gamma_{\mathbf{n}}\boldsymbol{v})$
$\mathrm{P}_{\mathcal{D}} :$	$u \mapsto \Pi_{\mathcal{D}} u$	$u \mapsto (\Pi_{\mathcal{D}} u, \mathbb{T}_{\mathcal{D}} u)$
$\mathrm{G}_{\mathcal{D}} :$	$u \mapsto \nabla_{\mathcal{D}} u$	$u \mapsto \nabla_{\mathcal{D}} u$
$\Vert u \Vert_{\mathcal{D}}$	$\Vert \nabla_{\mathcal{D}} u \Vert_{L^p}$	$\Vert \nabla_{\mathcal{D}} u \Vert_{L^p} + \vert \int_\Omega \Pi_{\mathcal{D}} u \vert$

- $\Omega \subset \mathbb{R}^3$
- $L = L^2(\Omega)^3$, so that $L' = L^2(\Omega)^3 = L$
- $\boldsymbol{L} = L^2(\Omega)^{3\times 3}$, so that $\boldsymbol{L}' = L^2(\Omega)^{3\times 3}$
- $\boldsymbol{W}_{\mathrm{D}} = H_{\mathrm{div}}(\Omega)^3$, and $V = \{0\}$
- $W_{\mathrm{G}} = H_0^1(\Omega)^3$

$$\mathrm{G} : H_0^1(\Omega)^3 \to L^2(\Omega)^{3\times 3} \text{ defined by } (\mathrm{G}u)_{i,j} = \frac{1}{2}(\partial_i u^{(j)} + \partial_j u^{(i)})$$

$$\mathrm{D} : H_{\mathrm{div}}(\Omega)^3 \to L^2(\Omega)^3 \text{ defined by } (\mathrm{D}\sigma)_i = \sum_{j=1}^{3} \partial_j \sigma^{(i,j)}$$

Hooke's law :

$$\boldsymbol{a}(\mathrm{G}u)_{i,j} = \lambda \sum_{k=1}^{3} (\mathrm{G}u)_{k,k}\delta_{i,j} + 2\mu(\mathrm{G}u)_{i,j}$$

with $\lambda \geq 0, \mu > 0$ (Lamé coefficients)

Equilibrium of a solid : find $u \in W_{\mathrm{G}}$ s.t. $-\mathrm{D}(\boldsymbol{a}(\mathrm{G}u)) = f$ with $f \in L'$

It's now time to conclude
Such that no results elude

The abstract Gradient Discretisation setting
Helps, in a variety of boundary conditions,
For easily formulating
General approximations.