Impact of robust discretisations on linear solvers

F. Hülsemann[1]

[1]EDF R&D

CIRM
May 2nd 2019

Acknowledgements

Projet ANR Fast4HHO (Contract ANR-17-CE23-0019)
Contributeurs:

- → J. Bonelle, Y. Fournier (EDF)
- → D. Di Pietro, F. Vilar (U. Montpellier),
- → D. Ruiz, P. Matalon (INP Toulouse),
- → C. Kruse, P. Mycek, U. Rüde (CERFACS)

plus, in addition:

- → D. Castanon (U. Montpellier),
- → R. Milani (EDF, ENPC),
- → G. Kemlin (EDF, ENPC),
- → N. Tardieu, N. Bereux (EDF), and
- → the *Code_Saturne* development team at EDF

Overview

→ Context: Computational PDEs at EDF

→ Robust discretizations in *Code_Saturne*: CDO, HHO

→ Convergence observations

- (scalar) diffusion
- Stokes

→ Conclusions, outlook

Context: Computational PDEs on complex geometries

→ Typical steps:
  - Represent the computational domain (generate a mesh)
  - Discretize the differential operators (chose the discretization schemes)
  - Chose the resolution strategy (e.g. linear solvers)

→ In general: **Strong** influence of the mesh quality on the precision of the computation

→ Complex geometries require large meshes (EDF: 10^9 cells in industrial instationary computations in 2017)

→ Generation of large, good quality meshes is **very** time consuming.

→ CFD consumes already a significant number of CPU hours. Increasing the computational requirements **is not an option**.

Context II: CFD at EDF

→ EDF has more than 30 years' of experience in the development of Computational PDE codes.

→ Current examples, both open source:

- Structural mechanics: code_aster (http://www.code-aster.org)
- CFD: *Code_Saturne* (https://www.code-saturne.org)

Code_Saturne in a few lines:

→ Originally: co-located finite volumes
→ Operator splitting (for the time being)
→ **Polyhedral, unstructured meshes**
→ Mostly single phase flows
→ Mostly implicit in time
→ Often incompressible
→ Up to 10^9 cells in industrial cases
→ Open source

→ Now also: CDO, HHO
→ First tests: stationary Stokes

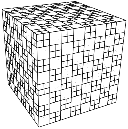
Context III: Robust Discretizations

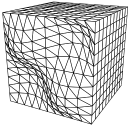
→ Roughly ten years ago: FVCA benchmark to test the ability of discretization schemes to deal with geometrically or topologically difficult meshes

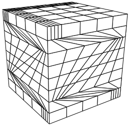
→ Since then: Publication of many new, robust schemes (dG, hdG, . . . , CDO, HHO)

→ However: There is no free lunch. Existing, fast linear solvers do not work well any more.

→ To the best of our knowledge, this statement still holds, in particular for higher order discretizations.







Robust discretizations in *Code_Saturne*

Motivation: Improve the quality of simulations on polyhedral or multi-elements meshes

Development of new discretization approaches

→ **Compatible Discrete Operator (CDO)** schemes at EDF/Ecole des Ponts

- Low order schemes
- Different families according to the location of DoF: CDO-Vb, CDO-VCb, CDO-Fb

→ **Hybrid High Order (HHO)** schemes at U. Montpellier/Ecole des Ponts

- Arbitrary order
- CDO-Fb $\leftrightarrow$ HHO $k = 0$

Linear solvers

For *Code_Saturne*, the aim is the Navier-Stokes operator. But let's start at the beginning with the **scalar diffusion equation**:

$$\nabla \cdot (K \nabla u) = f$$

with a "nice" tensor K (spd, no mean jumps, but not diagonal).

First observations:
M matrix property lost, even on regular meshes!
Positive off-diagonal entries too big to be ignored.

Tested solvers

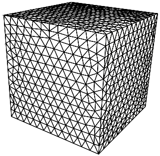
→ CG(BoomerAMG) via PETSc
→ k-cycle AMG (in-house implementation) – IPCG(k)
→ AGMG v3.2 for comparison

CDO Convergence example

Number of iterations of our in-house k-cycle AMG for the CDO face-based discretization.

FVCA test case TH

Unknowns	**Levels**	**IPCG(K)**	**PCG(Jacobi)**
5030	3	22	187
39184	5	24	366
309248	7	26	736
2457088	8	37	1463



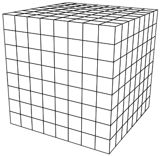
Results courtesy of Gaspard Kemlin.

HHO Convergence Examples

In-house k-cycle AMG (differs from AGMG in the treatment of positive off-diagnoals)

FVCA test case Hexa, k=1

Unknowns	**Levels**	**IPCG(K)**	**PCG(Jacobi)**
720	2	15	64
5184	3	21	131
39168	5	29	260
304128	6	42	517
2396160	7	77	1013



FVCA test case Hexa, k=2

Unknowns	**Levels**	**IPCG(K)**	**PCG(Jacobi)**
1440	2	59	207
10368	4	67	296
78336	4	67	461
608256	4	142	830

Results courtesy of Gaspard Kemlin.

Linear solvers: Lumping

One suggestion in the FE literature to deal with positive off-diagonal entries: Lumping onto the diagonal.
Test with CG preconditioned by *Cholesky factorisation* of the lumped matrix:

FVCA test case Hex

Number of CG iterations with lumped PC

Mesh	CDO	HHO, k=1	HHO, k=2
H03	5	21	81
H04	6	25	96
H08	7	46	135
H16	7	88	214
H32	7	out of mem.	out of mem.

Observation

Lumping:
for HHO: of no interest
for CDO: to be tested

Computations run on a 16GB desktop.

Linear solvers: Lumping for CDO

Changing the mesh: from hexahedra to tetrahedra

Number of CG iterations with lumped PC for CDO-fb:

FVCA test case Tetra		
Mesh	**Ndof**	**#it**
T2	8248	49
T3	16148	50
T4	31691	49
T5	62787	62
T6	124988	63

In all evidence, even for CDO, lumping is not the silver bullet.

Stokes/CDO

Note: Focus on face based discretization. Other CDO schemes are possible.

The CDO scheme is inf-sup stable $\Rightarrow$ no need for stabilization. The discrete Stokes operator takes hence the form

$$\left(\begin{array}{c|c} A & B^t \\ \hline B & 0 \end{array}\right)\left(\begin{array}{c} u \\ \hline p \end{array}\right) = \left(\begin{array}{c} f_u \\ \hline f_p \end{array}\right)$$

Solvers under investigation (Work in progress!):

- $\rightarrow$ Uzawa
- $\rightarrow$ artificial compressibility
- $\rightarrow$ Golub-Kahan bi-orthogonalization
- $\rightarrow$ *preconditioned GMRES on the full system (via PETSc)*

Saddle point preconditioners

Disclaimer: The literature on this topic is so rich that we do not claim to have tried out every possible method.

PC 1:

$$\left(\begin{array}{c|c} \tilde{A}^{-1} & B^t \\ \hline 0 & \tilde{S}^{-1} \end{array}\right) \quad \text{where} \quad \begin{array}{l} \tilde{A}^{-1}\text{: solving the } A \text{ system with BoomerAMG} \\ \tilde{S}^{-1} = -B(diagA)^{-1}B^t \end{array}$$

→ max. 4 GMRES iterations on a number of test cases

→ *Robust, but computationally costly*

Results courtesy of Jérôme Bonelle.

Saddle point preconditioners

PC 2:

$$\left(\begin{array}{c|c} \hat{A}^{-1} & 0 \\ \hline 0 & \mathrm{Id} \end{array}\right) \qquad \text{where} \qquad \hat{A}^{-1}\text{: 1 it. of BoomerAMG on the } A \text{ system}$$

Test case	**dim p**	**dim u**	**# it.**
PrG10	484	7464	193
PrG20	1764	26904	283
PrG30	3844	58344	290
PrG40	6724	101784	282
H16_2D	256	3168	55
H32_2D	1024	12480	132
H64_2D	4096	49536	132
H128_2D	16384	197376	143

Observations:

- $\rightarrow$ more iterations than PC 1
- $\rightarrow$ no h-independence
- $\rightarrow$ but, for these cases, still faster than PC 1

Results courtesy of Jérôme Bonelle.

Summary from a multigridder's point of view

Conclusion

Can one solve the CDO or HHO systems?	YES!
Can one solve these systems quickly?	Well ...

→ Loss of M-matrix property for the diffusion operator:

- Any future for point-wise smoothers (Gauß-Seidel, Jacobi)?
- What is a good *strength of connection measure* for these matrices?

→ None of the tested AMG implementations achieves h-independent convergence.

→ CDO: Existing multigrid solvers are not more than a stopgap solution (lack of robustness or speed or both).

→ HHO: Even for $k = 1$, fast *algebraic* solvers remain an open question.

Outlook

→ Beyond purely algebraic approaches: p-multigrid for HHO? (still requires a fast solver for $k = 0$, i.e. CDO)

→ AMG for Stokes (internship has just started)

→ The discretisation schemes are being extended to other differential operators (convection-diffusion, Navier-Stokes). More "surprises" ahead?

Summary from a multigridder's point of view II

Keep going! (That keeps LA in business.)

Thank you for your attention!