An Introduction to the theory of M-decompositions

Bernardo Cockburn

School of Mathematics
University of Minnesota

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Outline:

1. The HDG methods (2005 & 2009):
   - Jay Gopalakrishnan, Portland State University.
   - Raytcho Lazarov, Texas A&M University.
2. The first superconvergent HDG method (2008):
   - Bo Dong, University of Massachusetts Dartmouth.
   - Johnny Guzmán, Brown University.
3. Sufficient conditions for superconvergence (2012-13):
   - Weifeng Qiu, City University of Hong Kong.
   - Ke Shi, Old Dominion University.
4. Theory of M-decompositions (2016-17):
   - Guosheng Fu, Brown University.
   - Francisco-Javier Sayas, University of Delaware.
5. Ongoing work and references

The HDG methods. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

The model problem.

We want to numerically approximate the solutions of the following second-order elliptic model problem:

$$
\begin{aligned}
\mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{q} &= f && \text{in } \Omega,\\
\widehat{u} &= u_D && \text{on } \partial\Omega.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

The HDG methods.

The local solvers: A weak formulation on each element.

On the element $K \in \Omega_h$, given $\widehat{u}$ on ∂K and f, we have that $(\boldsymbol{q}, u)$ satisfies the equations

$$
\begin{aligned}
(\mathrm{c}\,\boldsymbol{q}, \boldsymbol{v})_K - (u, \nabla\cdot\boldsymbol{v})_K + \langle \widehat{u}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\
-(\boldsymbol{q}, \nabla w)_K + \langle \widehat{\boldsymbol{q}}\cdot\boldsymbol{n}, w\rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$, where

$$
\widehat{\boldsymbol{q}}\cdot\boldsymbol{n} = \boldsymbol{q}\cdot\boldsymbol{n} \qquad \text{on } \partial K.
$$

On the element $K \in \Omega_h$, we define $(\boldsymbol{q}_h, u_h)$ terms of $(\widehat{u}_h, f)$ as the element of $\boldsymbol{V}(K) \times W(K)$ such that

$$\begin{aligned} (\mathrm{c}\, \boldsymbol{q}_h, \boldsymbol{v})_K - (u_h, \nabla \cdot \boldsymbol{v})_K + \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\ -(\boldsymbol{q}_h, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial K} &= (f, w)_K, \end{aligned}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$, where (for the LDG-H method)

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau (u_h - \widehat{u}_h) \qquad \text{on } \partial K.$$

The HDG methods

The global problem: The weak formulation for $\widehat{u}_h$.

For each face $F \in \mathcal{E}_h^o$, we take $\widehat{u}_h|_F$ in the space $M(F)$. We determine $\widehat{u}_h$ by requiring that,

$$\langle \mu, [\![\widehat{\boldsymbol{q}}_h]\!]\rangle_F = 0 \quad \forall\, \mu \in M(F) \quad \text{if } F \in \mathcal{E}_h^o,$$
$$\widehat{u}_h = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

The HDG methods are generated by choosing the local spaces $\boldsymbol{V}(K)$, $W(K)$, $M(F)$ and the stabilization function τ.

The numerical traces of the LDG-H method are:

$$\widehat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-} [\![\boldsymbol{q}_h]\!],$$

$$\widehat{\boldsymbol{q}}_h = \frac{\tau^- {\boldsymbol{q}_h}^+ + \tau^+ {\boldsymbol{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+ \tau^-}{\tau^+ + \tau^-} [\![u_h]\!],$$

for $\boldsymbol{V}(K) := \boldsymbol{\mathcal{P}}_k(K)$, $W(K) := \mathcal{P}_k(K)$ and $M(F) := \mathcal{P}_k(F)$. So, the LDG-H method is a subset of the old DG methods (B.C and C.-W. Shu, SINUM, 98).

On general polyhedral elements, the LDG-H method

- For τ of order one, $\boldsymbol{q}_h$ converges with order $k + 1/2$ and u_h with order $k + 1$, for any $k \geq 0$.
- For τ of order $1/h$, $\boldsymbol{q}_h$ converges with order k and u_h with order $k + 1$, for any $k \geq 0$. True for $\boldsymbol{V}(K) := \boldsymbol{\mathcal{P}}_{k-1}(K)$.

(P.Castillo, B.C., I.Perugia and D.Shötzau, SINUM, 2000.)

The HDG methods.

Sufficient conditions for well possedness.

Theorem

The LDG-H approximation is well defined if, for each $K \in \Omega_h$,

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \boldsymbol{V}(K)$.

Theorem

The hybridized mixed method ($\tau \equiv 0$) *approximation is well defined if, for each* $K \in \Omega_h$,

- $\{\mu \in M(\partial K) : \langle \mu, 1\rangle_{\partial K} = 0\} = \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}(K), \nabla \cdot \boldsymbol{v} = 0\}$,
- $W(K) = \{\nabla \cdot \boldsymbol{V}(K) : \boldsymbol{v} \in \boldsymbol{V}(K), \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} \in \mathbb{R}\}$.

First superconvergent HDG methods.

The local postprocessing. (Nochetto and Gastaldi 88; Stenberg 88,91; B.C., Dong and Guzman, 08)

We seek HDG methods for which part of the error $u - u_h$, converge faster than the errors $u - u_h$ and $\boldsymbol{q} - \boldsymbol{q}_h$.

If this property holds, we introduce a new approximation $u_h^\star$. On each element K it lies in the space $W^\star(K) \supset W(K)$ and defined by

$$
\begin{aligned}
(\nabla {u_h}^\star, \nabla w)_K &= -(\mathrm{c}\boldsymbol{q}_h, \nabla w)_K && \text{for all } w \in W^\star(K)^\perp,\\
({u_h}^\star, \omega)_K &= (u_h, \omega)_K && \text{for all } w \in \widetilde{W}(K) \subset W(K).
\end{aligned}
$$

Then $u - u_h^\star$ will converge faster than $u - u_h$. This does happen for mixed methods!

First superconvergent HDG methods.

Method	$\boldsymbol{V}(K)$	$W(K)$	$M(F)$
RT	$\boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
BDM	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$

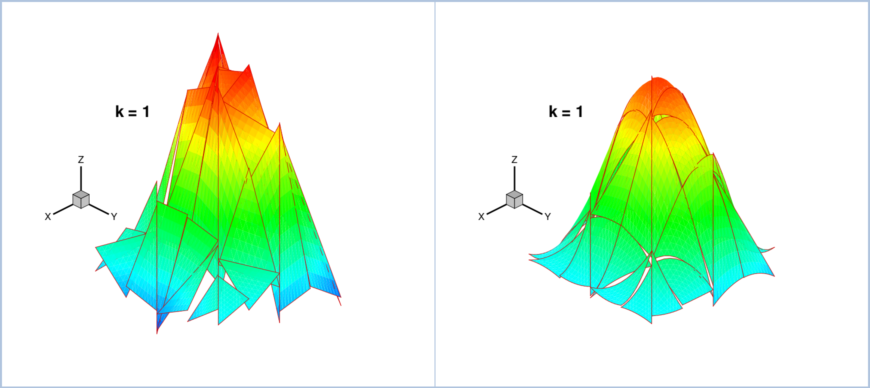
First superconvergent HDG methods. (B.C, B.Dong and J.Guzman, 08; B.C., J.Gopalakrishnan and F.-J. Sayas, 10)

The first superconvergent HDG method: the SFH method

Method	τ	$\boldsymbol{q}_h$	u_h	$\overline{u}_h$	k
RT	0	$k+1$	$k+1$	$k+2$	≥ 0
SFH	>0	$k+1$	$k+1$	$k+2$	≥ 1
LDG-H	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	≥ 1
BDM	0	$k+1$	k	$k+2$	≥ 2

First superconvergent HDG method.

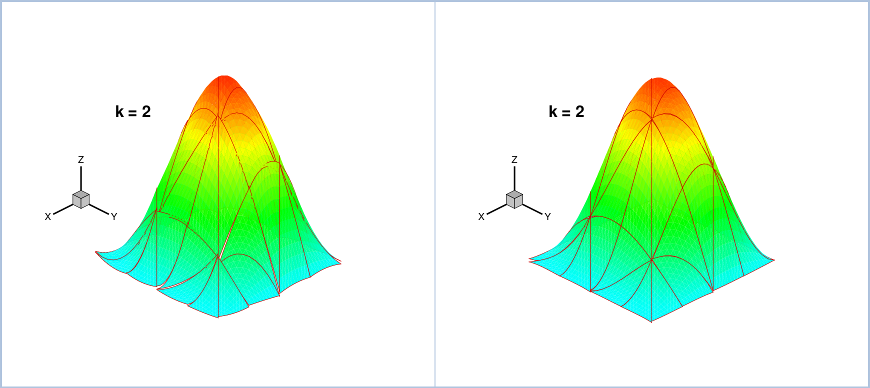
Illustration of the postprocessing. An HDG method for linear elasticity. (S.-C. Soon, B.C. and H. Stolarski, 2008.)



./figures/comparison between the approximate solution (left) and the post-processed solution (right) for linear polynomial approximations.

First superconvergent HDG method.

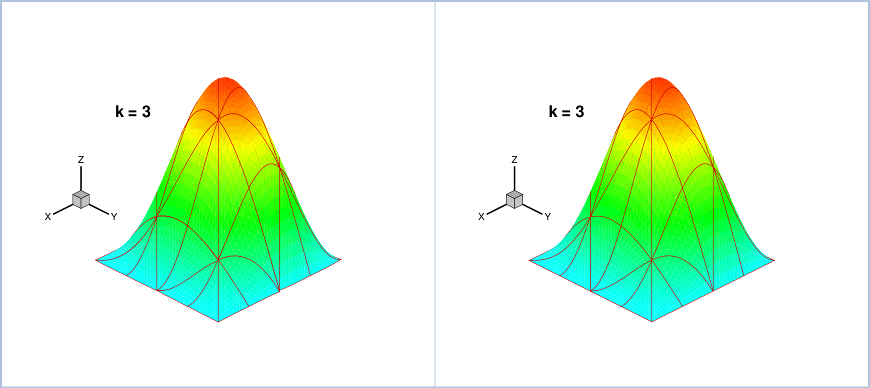
Illustration of the postprocessing. An HDG method for linear elasticity. (S.-C. Soon, B.C. and H. Stolarski, 2008.)



./figures/comparison between the approximate solution (left) and the post-processed solution (right) for quadratic polynomial approximations.

First superconvergent method.

Illustration of the postprocessing. An HDG method for linear elasticity. (S.-C. Soon, B.C. and H. Stolarski, 2008.)



./figures/comparison between the approximate solution (left) and the post-processed solution (right) for cubic polynomial approximations.

Sufficient conditions for superconvergence

The conditions on the local spaces. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

Theorem

Suppose that the local spaces are such that

$$
\begin{aligned}
& \boldsymbol{V}(K)\cdot\boldsymbol{n}+W(K) && \subset M(\partial K),\\
\mathcal{P}_0(K)\times\mathcal{P}_0(K)\subset\ & \nabla W(K)\times\nabla\cdot\boldsymbol{V}(K) && \subset \widetilde{\boldsymbol{V}}(K)\times\widetilde{W}(K),\\
& \widetilde{\boldsymbol{V}}^{\perp}\cdot\boldsymbol{n}\oplus\widetilde{W}^{\perp} && = M(\partial K).
\end{aligned}
$$

Then there is a stabilization function τ such that the HDG method superconverges.

Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k}(K) \times P^{k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}_3^k(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}^{Q}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}_2^k(K)$	$Q^k(K)$

Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a cube. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k,k}(K) \times P^{k,k+1,k}(K) \times P^{k,k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_7(K)$	$Q^k(K)$
$\mathbf{HDG}^Q_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_6(K)$	$Q^k(K)$

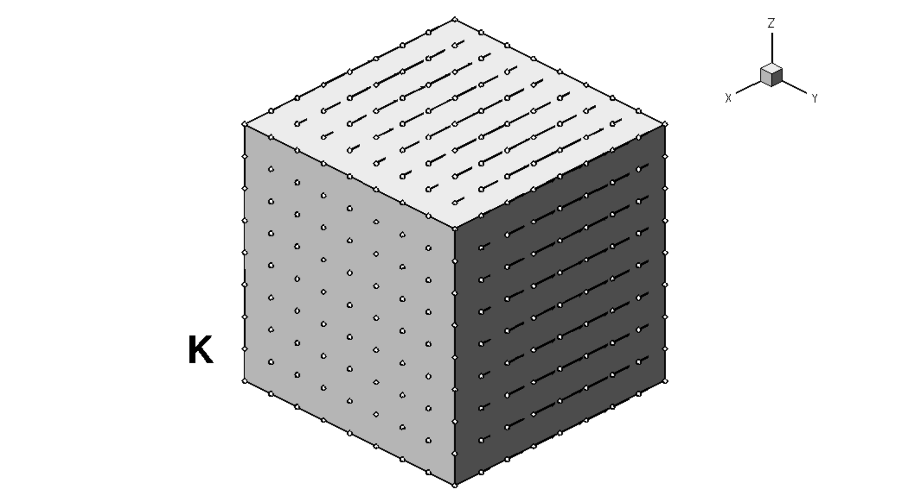
Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square or a cube. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	τ	$\|\boldsymbol{q} - \boldsymbol{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{RT}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{TNT}_{[k]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}^{Q}_{[k]}$	$\mathcal{O}(1) > 0$	$k+1$	$k+2$	$k+2$

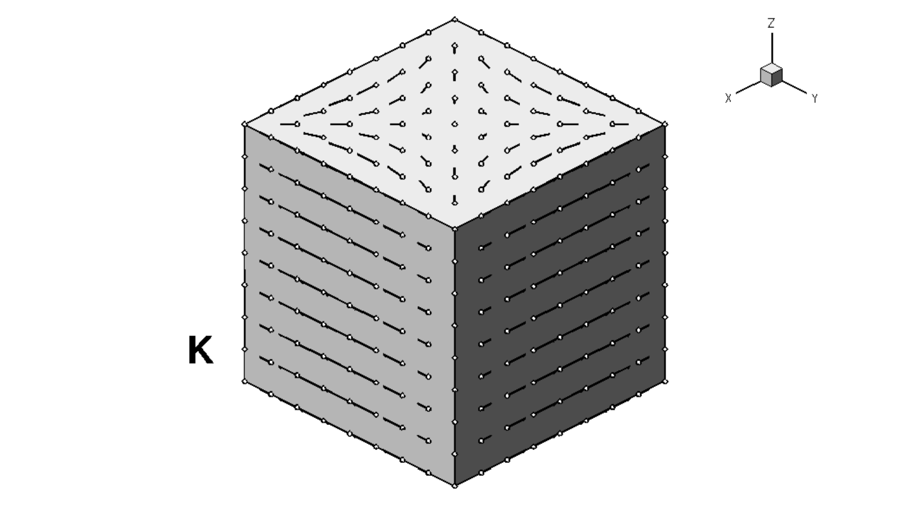
Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}^k_7(K)$.



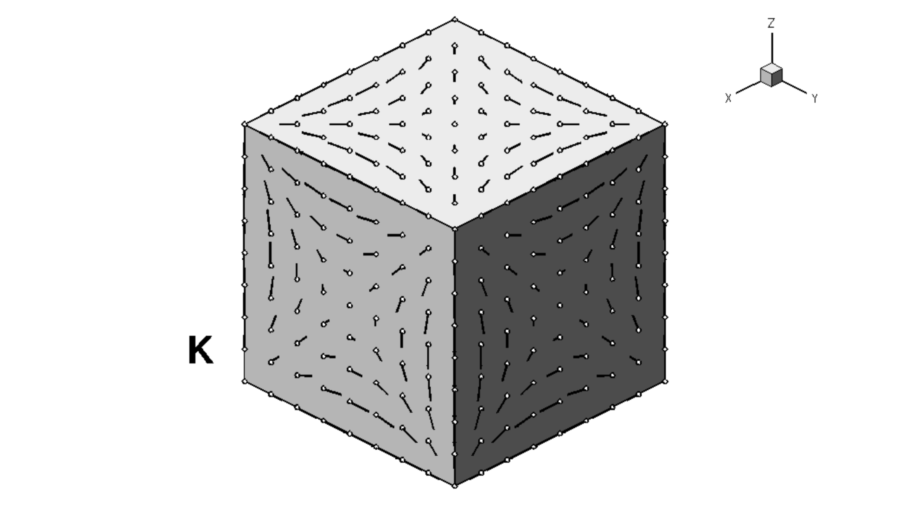
Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}_7^k(K)$.



Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}_7^k(K)$.



The theory of M-decompositions.

(B.C., G.Fu, F.-J. Sayas, Math. Comp., t2017; B.C. and G.Fu, 2D+3D, M^{2}AN, 2017)

Definition (The M-decomposition)

We say that $\boldsymbol{V} \times W$ admits an M-decomposition when

(a) $\mathrm{tr}(\boldsymbol{V} \times W) \subset M$,

and there exists a subspace $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ of $\boldsymbol{V} \times W$ satisfying

(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \widetilde{\boldsymbol{V}} \times \widetilde{W}$,

(c) $\mathrm{tr} : \widetilde{\boldsymbol{V}}^{\perp} \times \widetilde{W}^{\perp} \rightarrow M$ is an isomorphism.

Here $\widetilde{\boldsymbol{V}}^{\perp}$ and $\widetilde{W}^{\perp}$ are the $L^2(K)$-orthogonal complements of $\widetilde{\boldsymbol{V}}$ in $\boldsymbol{V}$, and of $\widetilde{W}$ in W, respectively.

The theory of M-decompositions.

A characterization of M-decompositions. (B.C., G.Fu, F.-J. Sayas, Math. Comp., 2017)

$$I_M(\boldsymbol{V}\times W) := \dim M - \dim\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}, \nabla\cdot\boldsymbol{v}=0\} - \dim\{w|_{\partial K} : w\in W, \nabla w = 0\}.$$

Theorem

For a given space of traces M, the space $\boldsymbol{V}\times W$ admits an M-decomposition if and only if

(a) $\mathrm{tr}(\boldsymbol{V}\times W)\subset M$,

(b) $\nabla W\times\nabla\cdot\boldsymbol{V}\subset\boldsymbol{V}\times W$,

(c) $I_M(\boldsymbol{V}\times W)=0$.

In this case, we have

$$M = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}, \nabla\cdot\boldsymbol{v}=0\}\oplus\{w|_{\partial K} : w\in W, \nabla w=0\},$$

where the sum is orthogonal.

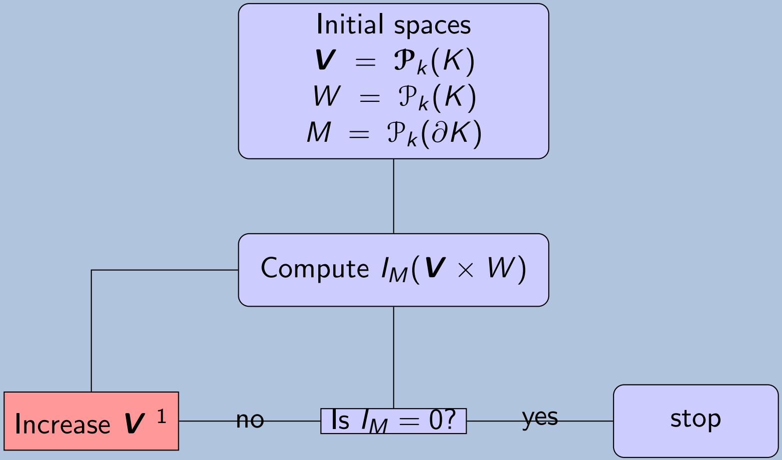
Table: Construction of spaces $\boldsymbol{V} \times W$ admitting an M-decomposition, where the space of traces $M(\partial K)$ includes the constants. The given space $\boldsymbol{V}_g \times W_g$ satisfies the inclusion properties (a) and (b).

$\boldsymbol{V}$	W		$\nabla \cdot \boldsymbol{V}$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\mathrm{fillM}} \oplus \delta \boldsymbol{V}_{\mathrm{fillW}}$	W_g	(if $\supset \mathcal{P}_0(K)$)	$= W_g$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\mathrm{fillM}}$	W_g	(if $\supset \mathcal{P}_0(K)$)	$\subset W_g$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\mathrm{fillM}}$	$\nabla \cdot \boldsymbol{V}_g$	(if $\supset \mathcal{P}_0(K)$)	$= \nabla \cdot \boldsymbol{V}_g$

$\delta \boldsymbol{V}$	$\nabla \cdot \delta \boldsymbol{V}$	$\gamma \delta \boldsymbol{V}$	$\dim \delta \boldsymbol{V}$
$\delta \boldsymbol{V}_{\mathrm{fillM}}$	$\{\boldsymbol{0}\}$	$\subset M, \cap \gamma \boldsymbol{V}_{g_s} = \{0\}$	$I_M(\boldsymbol{V}_g \times W_g)$
$\delta \boldsymbol{V}_{\mathrm{fillW}}$	$\subset W_g, \cap \nabla \cdot \boldsymbol{V}_g = \{0\}$	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$

The theory of M-decompositions.

A flowchart to construct M-decompositions



[1] such that properties (a) and (b) of an M-decomposition are not violated and the M-index is decreased.

Construction of M-decompositions

Theorem

Let $\boldsymbol{V}_g \times W_g$ *satisfy properties* (a) *and* (b) *of an* M*-decomposition. Assume that* $\delta \boldsymbol{V}_{\text{fillM}}$ *satisfies the following hypotheses:*

(a) $\nabla \cdot \delta \boldsymbol{V}_{\text{fillM}} = \{0\}$,

(b) $\delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K} \subset M$,

(c) $\delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K}$ *and* $\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \ \boldsymbol{v} \in \boldsymbol{V}, \ \nabla \cdot \boldsymbol{v} = 0\}$ *are linearly independent,*

(d) $\dim \delta \boldsymbol{V}_{\text{fillM}} = \dim \delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K} = I_M(\boldsymbol{V}_g \times W_g)$

Then, $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}) \times W_g$ *admits an* M*-decomposition.*

A construction of M-decompositions

A three-step procedure to construct the filling space $\delta \boldsymbol{V}_{\mathrm{fillM}}$

(1) Characterize the trace space $\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \ \boldsymbol{v}\in\boldsymbol{V},\ \nabla\cdot\boldsymbol{v}=0\}$

(2) Find a trace space $C_M \subset M(\partial K)$ such that

$$C_M \oplus \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}, \nabla\cdot\boldsymbol{v}=0\} = \{\mu\in M : \langle\mu,1\rangle_{\partial K}=0\}$$

note that the dimension of the space C_M is equal to $I_M(\boldsymbol{V}\times W)$

(3) Set $\delta \boldsymbol{V}_{\mathrm{fillM}} := \{\boldsymbol{v}_\mu : \ \mu\in C_M\}$, where $\boldsymbol{v}_\mu$ is divergence-free function such that $\boldsymbol{v}_\mu\cdot\boldsymbol{n}|_{\partial K}=\mu$

A construction of M-decompositions

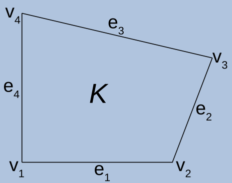
$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K)$$

2D element	$I_M(\boldsymbol{V} \times W)$	3D element	$I_M(\boldsymbol{V} \times W)$
triangle	0 $(k \geq 0)$	tetrahedron	0 $(k \geq 0)$
quadrilateral	1 $(k=0)$, 2 $(k \geq 1)$	pyramid	1 $(k=0)$, 3 $(k \geq 1)$
pentagon	2 $(k=0)$, 4 $(k=1)$, 5 $(k \geq 2)$	prism[2]	1 $(k=0)$, 3 $(k \geq 1)$
hexagon	3 $(k=0)$, 6 $(k=1)$, 8 $(k=2)$, 9 $(k \geq 3)$	hexahedron[2]	2 $(k=0)$, 6 $(k=1)$, 9 $(k \geq 2)$

[2] no parallel faces

A construction of M-decompositions

An example of $\delta \boldsymbol{V}_{\mathrm{fillM}}$ on a quadrilateral



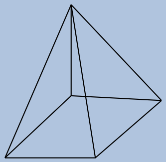
$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),$$

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \mathrm{span}\{\nabla\times(\xi_4 \lambda_4^k), \nabla\times(\xi_4 \lambda_3^k)\}.$$

- λ_i is a linear function that vanishes on edge e_i.
- $\xi_4 \in H^1(K)$ is a function such that its trace on each edge is linear and vanishes at the vertices v_1, v_2, and v_3.

A construction of M-decompositions

An example of $\delta \boldsymbol{V}_{\mathrm{fillM}}$ on the reference pyramid



$$K := \{(x, y, z) : 0 < x, 0 < y, 0 < z, x + z < 1, y + z < 1\}$$

$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K)$$

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \begin{cases} \mathrm{span}\{\nabla\times(\frac{x\,y}{1-z}\nabla z)\} & \text{if } k = 0 \\ \mathrm{span}\{\nabla\times(\frac{x\,y^{k+1}}{1-z}\nabla z), \nabla\times(\frac{y\,x^{k+1}}{1-z}\nabla z), \nabla\times(\frac{x\,y}{1-z}\nabla x)\} & \text{if } k \geq 1 \end{cases}$$

A construction of M-decompositions.

From M-decompositions to hybridized mixed methods

Theorem

Let the space $\boldsymbol{V} \times W$ *admit an* M*-decomposition and assume that* $\nabla \cdot \boldsymbol{V}_g \subsetneq W$. *Then,*

$$\boldsymbol{V} \times \nabla \cdot \boldsymbol{V} \text{ admits an } M\text{-decomposition.}$$

Moreover, let $\delta \boldsymbol{V}_{\mathrm{fillW}}$ *satisfy the following hypotheses:*

(a) $\delta \boldsymbol{V}_{\mathrm{fillW}} \cdot \boldsymbol{n}|_{\partial K} \subset M$,

(b) $\nabla \cdot \delta \boldsymbol{V}_{\mathrm{fillW}} \oplus \nabla \cdot \boldsymbol{V} = W_g$,

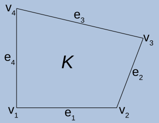
(c) $\dim \delta \boldsymbol{V}_{\mathrm{fillW}} = \dim \nabla \cdot \delta \boldsymbol{V}_{\mathrm{fillW}}$,

Then $(\boldsymbol{V} \oplus \delta \boldsymbol{V}_{\mathrm{fillW}}) \times W$ *admits an* M*-decomposition.*

For the above choices of spaces, we can set stabilization operator $\tau = 0$ in and obtain hybridized mixed methods.

A construction of M-decompositions

Spaces for hybridized mixed methods on a quadrilateral



$$\boldsymbol{V}^{hdg} \times W^{hdg} \times M := \boldsymbol{\mathcal{P}}_k(K) \oplus \delta \boldsymbol{V}_{\text{fillM}} \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),$$

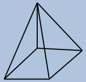
$$\delta \boldsymbol{V}_{\text{fillM}} := \text{span}\{\nabla\times(\xi_4 \lambda_4^k), \nabla\times(\xi_4 \lambda_3^k)\}.$$

$$\delta \boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\, \widetilde{\mathcal{P}}_k(K).$$

	$\boldsymbol{V}$	W	M	τ
UMX	$\boldsymbol{V}^{hdg} \oplus \delta \boldsymbol{V}_{\text{fillW}}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	0
HDG	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	> 0
LMX	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(\partial K)$	0

A construction of M-decompositions

Spaces for hybridized mixed method on a pyramid



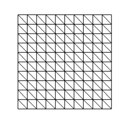
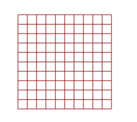
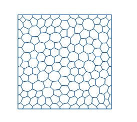
$$\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}} \times M := \boldsymbol{\mathcal{P}}_k(K) \oplus \delta \boldsymbol{V}_{\mathrm{fillM}} \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),\ k \geq 1$$

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \mathrm{span}\{\nabla\times(\frac{x\,y^{k+1}}{1-z}\nabla z),\ \nabla\times(\frac{y\,x^{k+1}}{1-z}\nabla z),\ \nabla\times(\frac{x\,y}{1-z}\nabla x)\}.$$

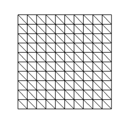
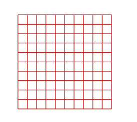
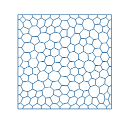
$$\delta \boldsymbol{V}_{\mathrm{fillW}} := \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K).$$

	$\boldsymbol{V}$	W	M	τ
UMX	$\boldsymbol{V}^{hdg} \oplus \delta \boldsymbol{V}_{\mathrm{fillW}}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	0
HDG	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	> 0
LMX	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(\partial K)$	0

History of convergence of LDG-H with $k = 1$

						
h	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate
	$\tau = 1$					
0.1	0.15E-2	-	0.83E-2	-	0.52E-2	-
0.05	0.18E-3	3.06	0.16E-2	2.36	0.10E-2	2.34
0.025	0.23E-4	3.03	0.28E-3	2.52	0.19E-3	2.43
0.0125	0.28E-5	3.02	0.44E-4	2.68	0.35E-4	2.46

History of convergence of M-decompositions with $k = 1$

						
h	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate
	$\tau = 1$					
0.1	0.15E-2	-	0.26E-2	-	0.17E-2	-
0.05	0.18E-3	3.06	0.31E-3	3.06	0.21E-3	3.02
0.025	0.23E-4	3.03	0.38E-4	3.03	0.27E-4	2.95
0.0125	0.28E-5	3.02	0.47E-5	3.02	0.35E-5	2.96

The theory of M-decompositions

Provides:

1. A systematic way of constructing superconvergent HDG and hybridized mixed methods for elements of arbitrary shapes.
2. A systematic approach to satisfying elementwise inf-sup conditions, stabilized (HDG) or not (mixed methods).
3. A systematic way of constructing finite element commuting diagrams.

References and ongoing work.

Work (with Guosheng Fu).

1. Superconvergence by M-decompositions. Part I: general theory for HDG methods for diffusion. With F.-J. Sayas. Math. Comp., 2017.
2. Superconvergence by M-decompositions. Part II: Construction of two-dimensional finite elements. M2AN, 2017.
3. Superconvergence by M-decompositions. Part III: Construction of three-dimensional finite elements. M2AN, 2017.
4. A note on the devising of superconvergent HDG methods for the Stokes flow by M-decompositions. With W. Qui. IMA, 2017.
5. Devising superconvergent HDG methods with symmetric approximate stresses for linear elasticity. IMA, 2018.
6. A systematic construction of finite element commuting exact sequences. SINUM, 2017.

References and ongoing work.

Ongoing work.

1. Automatic generation of the local spaces.
2. Incompressible Navier-Stokes.
3. 3D elasticity with symmetric stresses.
4. Maxwell equations.
5. The biharmonic, plates.