On pressure-robust space discretizations and incompressible high Reynolds number flows

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H. Helmholtz

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Coauthors

- Christian Merdon, Weierstrass Institute (WIAS)
- Leo Rebholz, U Clemson
- Philipp W. Schroeder, U Göttingen (special thanks !!!)
- Nicolas Gauger, TU Kaiserslautern

Special thanks to J. Schöberl, C. Lehrenfeld: NGSOLVE

Main references

improved understanding of steady Stokes & beyond

V. John, A. L., C. Merdon, M. Neilan, L. Rebholz: On the divergence constraint in mixed FEM for incompressible flows. SIAM Review, Vol. 59(3), 2017.

N. Gauger, P. Schroeder, A. Linke: On high-order pressure-robust space discretisations, their advantages for incompressible high Reynolds number generalised Beltrami flows and beyond. arXiv 1808.10711.

improved understanding of (laminar) transient high Reynolds number Navier-Stokes

Outline

- 3 examples: pressure-robust vs. non-pressure-robust solvers
- original sin of incompressible CFD: a relaxed $\mathbf{L}^2$-orthogonality
- 2 model problems
- numerical analysis
- classification of pressure-robust CFD solvers
- connection to vortex-dominated flows

pressure-robustness inside:

new seal of quality for incompressible/low Mach number CFD

Incompressible Navier-Stokes equations (iNSE)

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

Focus:

- iNSE in primitive variables
- space discretization at high Reynolds numbers, i.e., $0 < \nu \ll 1$

A warning – and a promise

Connection to POEMs:

- focus not on polyhedral meshes
- bridging pure and numerical mathematics

Example 1: Moving Gresho vortex

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$

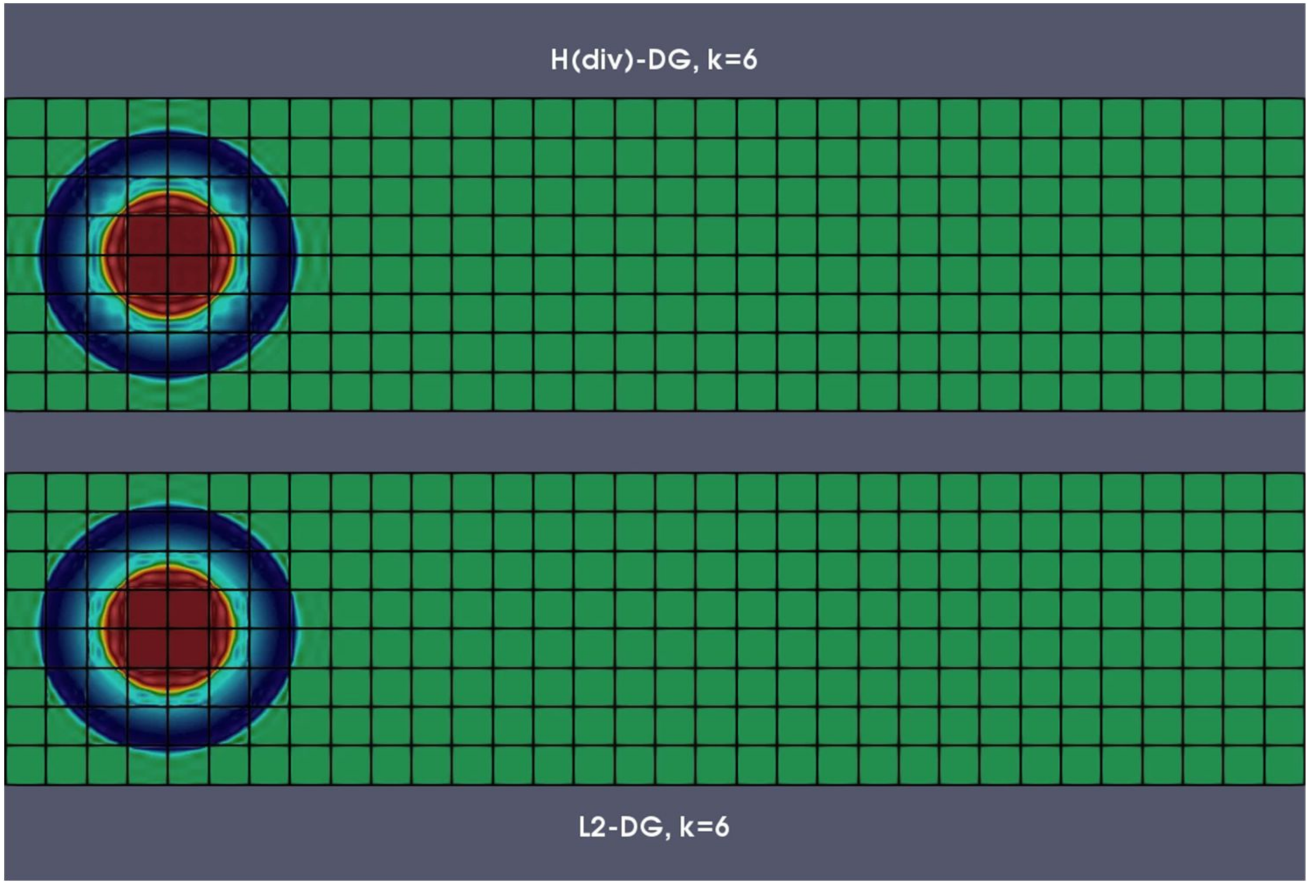
$$\nabla \cdot \mathbf{u} = 0$$

$$\nu = 10^{-5}, \qquad t \in (0, 15]$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 1: Moving Gresho vortex

$$\mathbb{BDM}_6 - \mathbb{P}_5^{\mathrm{dc}} + \text{upwind} \quad \text{vs.} \quad \mathbb{P}_6^{\mathrm{dc}} - \mathbb{P}_5^{\mathrm{dc}} + \text{upwind}$$



Example 1: Moving Gresho vortex

$$\underbrace{\mathbb{BDM}_6 - \mathbb{P}_5^{\mathrm{dc}}}_{\text{pressure-robust}} \quad \text{vs.} \quad \mathbb{P}_6^{\mathrm{dc}} - \mathbb{P}_5^{\mathrm{dc}}$$

References (Philipp W. Schroeder):

- PhD thesis, U Göttingen, 2019.
- www.youtube.com/watch?v=wrZTUrGxVSc

Why pressure-robust DG method more accurate?

A warning

- talk not about mass conservation: velocity trial functions
- but pressure-robustness: velocity test functions
- confusion in Galerkin setting: trial functions = test functions

special thanks to R. Eymard !!!

Reference: A. Linke, C. Merdon: Pressure-robustness […]. CMAME 2016.

Example 2: Planar lattice flow

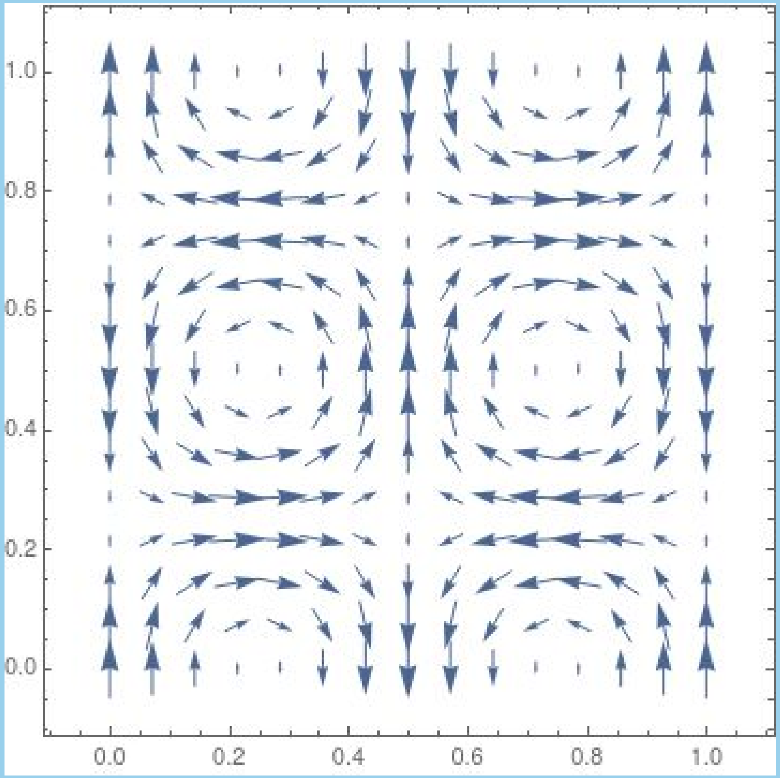
$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nu = 10^{-5}, \qquad t \in (0, 10]$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 2: Planar lattice flow



$$\mathbf{u}_0(\mathbf{x}) = \begin{pmatrix} \sin(2\pi x)\sin(2\pi y) \\ \cos(2\pi x)\cos(2\pi y) \end{pmatrix},$$

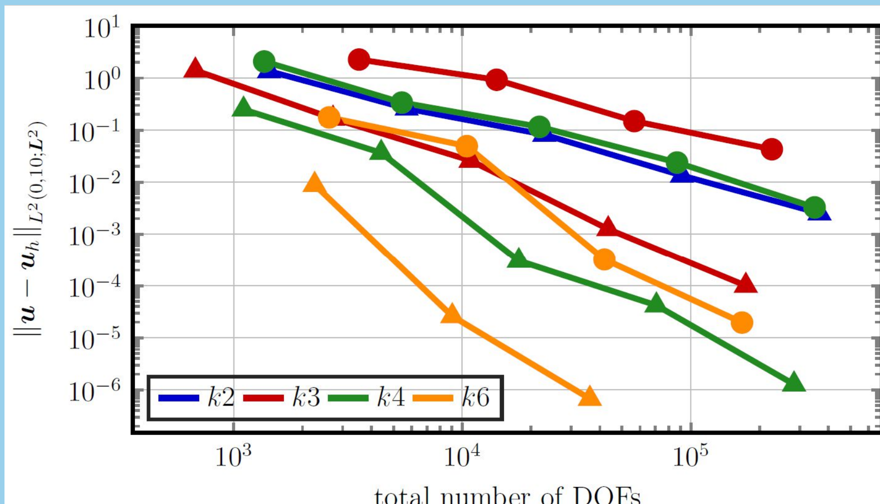
$$\mathbf{u}(t,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})e^{-8\pi^2 \nu t}$$

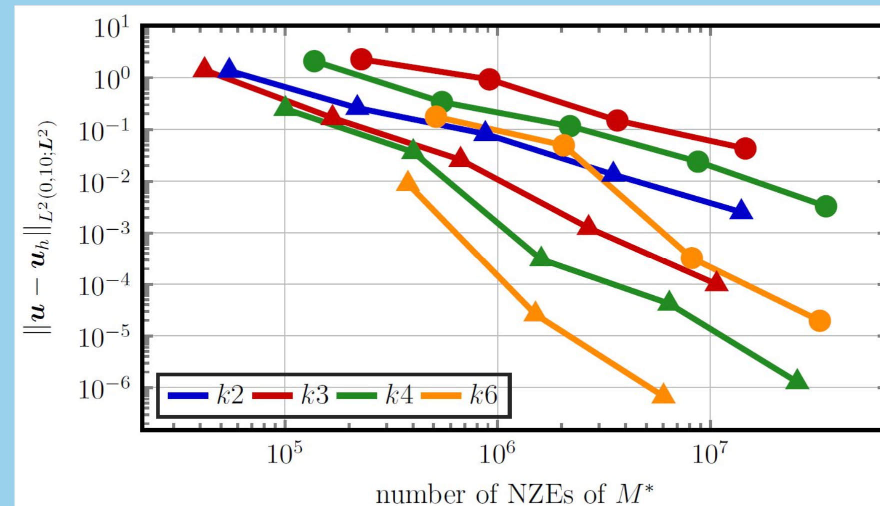
$$p_0(\mathbf{x}) = \frac{1}{4}\left(\cos(4\pi x) - \cos(4\pi y)\right),$$

$$p(t,\mathbf{x}) = p_0(\mathbf{x})e^{-16\pi^2 \nu t}$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 2: Planar lattice flow





$$\underbrace{\mathbb{BDM}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}}_{\text{pressure-robust}} \quad \text{vs.} \quad \mathbb{P}^{\mathrm{dc}}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}$$

- pressure-robust solvers (triangles) outperform non-pressure-robust ones (circles)
- coarse grids: non-pressure-robust solvers lose half of (formal) convergence order

Example 3: Steady Stokes flow

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla\cdot\mathbf{u} = 0$$

$$\boxed{\nu = 10^{-3}, \mathbf{u} \in \mathbf{H}_0^1}$$

- nontrivial forcing f
- small viscosity

Example 3: Steady Stokes flow

$$\xi = x^2(1-x)^2 y^2 (1-y)^2$$
$$\mathbf{u} = \mathbf{curl}\, \xi$$
$$p = x^3 + y^3 - \frac{1}{2}$$

$$\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p$$

- small viscosity
- manufactured f: nearly gradient field

Classical CR-FEM	Pressure-robust CR-RT_0-FEM

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On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime

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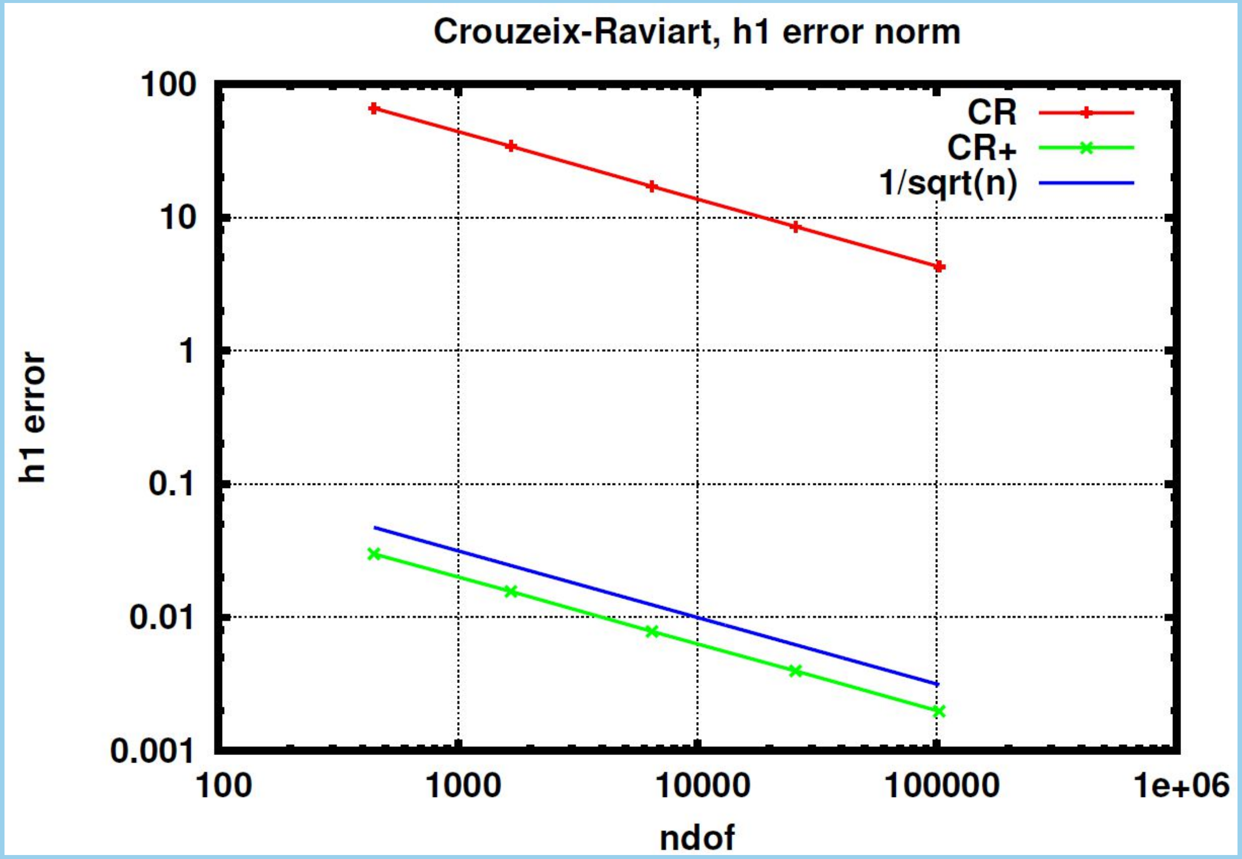
$$\int \mathbf{f} \cdot \mathbf{v}_h \, \mathrm{dx}$$

classical discrete forcing

$$\int \mathbf{f} \cdot I_h^{\mathrm{RT}_0} \mathbf{v}_h \, \mathrm{dx}$$

Raviart-Thomas standard interpolator applied elementwise to CR velocity test function

Example 3: Steady Stokes flow



- classical CR-FEM vs. pressure-robust CR-RT_0-FEM
- pressure-robust gain: 10 refinement levels

Example 3: Steady Stokes flow

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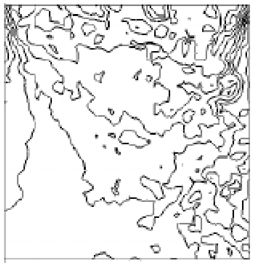
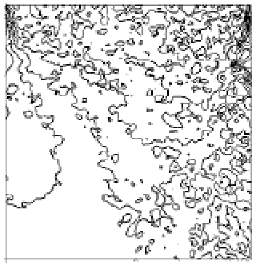
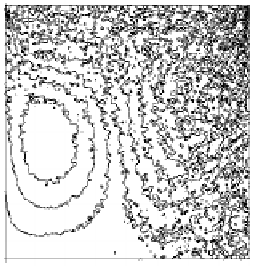
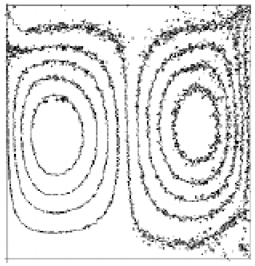
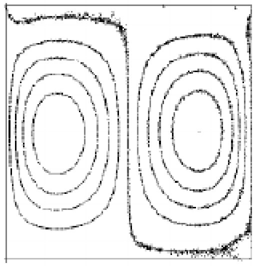
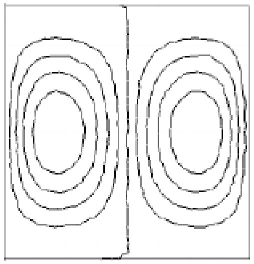
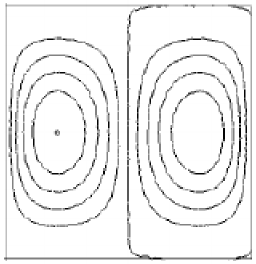
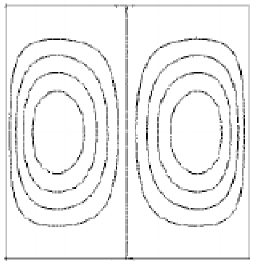
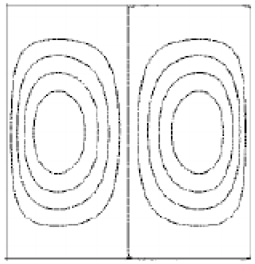
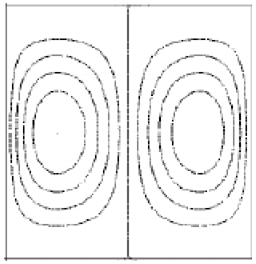
CR					
CR^+					

- new understanding: CR-FEM forcing too strong
- accurate discrete $\mathbf{L}^2$ $-$scalar product (test functions) required

Pressure-robustness vs. non-pressure-robustness

- How to explain dramatic superior accuracy of pressure-robust methods?
- Common reason behind?

Original sin of incompressible/low Mach number CFD

Relaxation of divergence constraint in

- discretely inf-sup stable mixed Stokes methods
- pressure-stabilized mixed Stokes methods

hidden consistency error

relaxed $\mathbf{L}^2$-orthogonality of arbitrary gradient vs. discretely divergence-free velocity test functions

Connection to some open questions in CFD

- zoo of stabilizations (grad-div stabilization, SUPG, …)
- structure-preserving low order vs. high-order
- MAC scheme: structured vs. unstructured meshes
- significance of divergence-free mixed FEMs
- turbulence mixing vs. coherent structures

Two model problems

Model problem 1:

$$\begin{aligned} \mathbf{u}_t + \nabla p &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Model problem 2:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Starting point for understanding high Reynolds number case:

- time-dependent model problems (not steady Stokes)
- Helmholtz-Hodge projector

Helmholtz-Hodge projector

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \int \mathbf{v} \cdot \nabla \phi \, \mathrm{dx} = 0, \quad \text{for all } \phi \in H^1\}!$$

$$\boxed{\mathbf{L}^2 = \mathbf{L}^2_\sigma \bigoplus_{\mathbf{L}^2} \nabla(H^1)}$$

- $\mathbf{L}^2_\sigma$: $\mathbf{L}^2$-orthogonal complement to $\mathbf{L}^2$ gradient fields
- major importance in pure mathematics
- key for understanding pressure-robustness

Helmholtz-Hodge projector

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \quad \underbrace{-\int \mathbf{v} \cdot \nabla \phi \, \mathrm{dx} = 0}_{\text{distributional divergence for } \phi \in C_0^\infty}, \quad \text{for all } \phi \in H^1\}!$$

Properties:

- $\mathbf{L}^2_\sigma \subset \mathbf{H}(\mathrm{div})$: test functions with compact support
- $\mathbf{L}^2_\sigma$: weakly divergence-free vector fields
- smooth divergence-free vector fields with compact support
- topological closure: elements weakly divergence-free & vanishing normal component at boundary
- contains divergence-free BDM-FEM & RT-FEM

Helmholtz-Hodge projector

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \underbrace{-\int \mathbf{v} \cdot \nabla \phi \, \mathrm{dx} = 0}_{\text{distributional divergence for } \phi \in C_0^\infty}, \quad \text{for all } \phi \in H^1\}!$$

Key for pressure-robustness:

divergence-free BDM & divergence-free RT vector fields (with vanishing normal boundary component)

$\mathbf{L}^2$-orthogonal to arbitrary gradient fields !!!

Thanks to F. Brezzi, D. Marini, J. Douglas, P.-A. Raviart, J.-M. Thomas, …

Helmholtz-Hodge projector

$$\mathbf{L}^2 = \mathbf{L}^2_\sigma \oplus_{\mathbf{L}^2} \nabla(H^1)$$

Helmholtz-Hodge decomposition –
fundamental theorem of vector calculus

Helmholtz-Hodge projector

Helmholtz-Hodge projector $\mathbb{P}(\mathbf{f})$: $\mathbf{L}^2$ projector onto $\mathbf{L}^2_\sigma$

Helmholtz-Hodge projector

Fundamental property ($\mathbf{L}^2$-orthogonality):

$$\mathbb{P}(\nabla \phi) = \mathbf{0}$$

Helmholtz-Hodge projector: related to curl operator

Model problem 1

$$\mathbf{u}_t + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0$$

$$\mathbf{u} \in L^1(0, T; \mathbf{H}_0(\mathrm{div}))$$
$$p \in L^1(0, T; L_0^2)$$

$$\mathbf{f} \in L^1(0, T; \mathbf{L}^2)$$
$$\mathbf{u}_0 \in \mathbf{L}_\sigma^2$$

model setting

Model problem 1

$$\mathbb{P}(\mathbf{u}_t) = \mathbb{P}(\mathbf{f}) \qquad \overset{\mathbf{u}\in\mathbf{L}^2_\sigma}{\Rightarrow}$$

$$\mathbf{u}_t = \mathbb{P}(\mathbf{f}) \qquad \Rightarrow$$

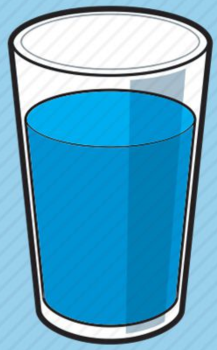
$$\boxed{\mathbf{u}(T) = \mathbf{u}_0 + \int_0^T \mathbb{P}(\mathbf{f})\,\mathrm{dt}}$$

Divergence-constraint:

- **u** dependent on $\mathbb{P}(\mathbf{f})$
- equivalence classes of forces involved – and a semi-norm

Velocity-equivalence of forces

hydrostatics



$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nabla \phi$$

$$\nabla \phi \simeq \mathbf{0}$$

$$\mathbf{f} \simeq \mathbf{g} \quad \Leftrightarrow$$

$$\mathbb{P}(\mathbf{f}) = \mathbb{P}(\mathbf{g}) \quad \Leftrightarrow$$

$$\exists_{\nabla \phi} : \mathbf{f} = \mathbf{g} + \nabla \phi$$

velocity-equivalence induced by semi-norm $\|\mathbb{P}(\mathbf{f})\|_{\mathbf{L}^2}$

Model problem 1 & conforming Galerkin mixed methods

$$\mathbf{H}_0(\mathrm{div}) \to \mathbf{V}_h \subset \mathbf{H}_0(\mathrm{div})$$

$$L^2_0 \to Q_h \subset L^2_0$$

Let‘s assume inf-sup-condition for pair: $(\mathbf{V}_h, Q_h)$

$$\inf_{q_h \in Q_h \neq 0} \sup_{\mathbf{v}_h \in \mathbf{V}_h \neq \mathbf{0}} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|q\|_{L^2} \|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div})}} \geq \beta_h \geq \bar{\beta} > 0$$

yields well-posed discrete mixed Galerkin approximation

Model problem 1 & conforming Galerkin mixed methods

$$\mathbf{L}^2_\sigma \to \mathbf{L}^2_{\sigma,h} \qquad \mathbb{P} \to \mathbb{P}_h$$

Implicitly defined discretely divergence-free vector fields

$$\mathbf{L}^2_{\sigma,h} := \{\mathbf{v}_h \in \mathbf{V}_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0 \quad \text{for all } q_h \in Q_h\}$$

Discrete Helmholtz-Hodge projector:

$\mathbf{L}^2$-projection $\mathbb{P}_h$ onto $\mathbf{L}^2_{\sigma,h}$

Model problem 1

$$\mathbf{u}_{0,h} := \mathbb{P}_h(\mathbf{u}_0)$$

$$\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}_0) + \int_0^T \mathbb{P}_h(\mathbf{f})\,\mathrm{dt}$$

Galerkin discretization

$$\mathbb{P} \to \mathbb{P}_h$$

Model problem 1

$$\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}_0) + \int_0^T \mathbb{P}_h(\mathbf{f})\,\mathrm{dt}$$

$$= \mathbb{P}_h(\mathbf{u}_0) + \int_0^T \left(\mathbb{P}_h(\mathbf{u}_t) + \mathbb{P}_h(\nabla p)\right)\,\mathrm{dt}$$

$$\boxed{\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}(T)) + \int_0^T \mathbb{P}_h(\nabla p)\,\mathrm{dt}}$$

inf-sup stability: optimality for $\mathbb{P}_h(\mathbf{u}(T))$

Model problem 1

$$\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}(T)) + \int_0^T \mathbb{P}_h(\nabla p)\,\mathrm{dt} \qquad \Rightarrow$$

$$\|\mathbf{u}_h(T) - \mathbb{P}_h(\mathbf{u}(T))\|_{\mathbf{L}^2} \leq \int_0^T \underbrace{\|\mathbb{P}_h(\nabla p)\|_{\mathbf{L}^2}}_{???}\,\mathrm{dt}$$

- inf-sup stability: optimality for $\mathbb{P}_h(\mathbf{u}(T))$
- what about for $\|\mathbb{P}_h(\nabla p)\|_{L^2}$???

Non-pressure-robust	Pressure-robust
Example (Taylor-Hood):	Example (Brezzi-Douglas-Marini):
$\boldsymbol{\mathbb{P}}_k - \mathbb{P}_{k-1}$	$\mathbb{BDM}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}$
$\nabla \cdot (\boldsymbol{\mathbb{P}}_k) \supset \mathbb{P}_{k-1}$	$\nabla \cdot (\mathbb{BDM}_k) = \mathbb{P}^{\mathrm{dc}}_{k-1}$
$\mathbf{L}^2_{\sigma,h} \not\subset \mathbf{L}^2_\sigma$	$\mathbf{L}^2_{\sigma,h} \subset \mathbf{L}^2_\sigma$
$\underbrace{\mathbb{P}_h(\nabla p)}_{\text{hidden consistency error}\neq \mathbf{0}}$	$\mathbb{P}_h(\nabla p) = \mathbf{0}$

Non-pressure-robust

Example (Taylor-Hood, …):

$$\mathbb{P}_h(\nabla p) \neq \mathbf{0}$$

$$\mathbf{u}_h(T) = \mathbf{u}_{0,h} + \int_0^T \mathbb{P}_h(\mathbf{f})\,\mathrm{dt}$$

$$\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}(T)) + \int_0^T \mathbb{P}_h(\nabla p)\,\mathrm{dt}$$

- $\mathbf{u}_h = \mathbf{u}_h(\mathbf{u}, p, T)$: pressure-dependent
- $\mathbf{u}_h = \mathbf{u}_h(\mathbf{f})$: data-dependence inconsistent with iNSE
- error: arbitrarily large for long times T

Pressure-robust

Example (Brezzi-Douglas-Marini, …):

$$\mathbb{P}_h(\nabla p) = \mathbf{0}$$

$$\mathbf{u}_h(T) = \mathbf{u}_{0,h} + \int_0^T \mathbb{P}_h(\mathbb{P}(\mathbf{f}))\,\mathrm{dt}$$

$$\mathbf{u}_h(T) = \mathbb{P}_h(\mathbf{u}(T))$$

- $\mathbf{u}_h = \mathbf{u}_h(\mathbf{u})$: pressure-robust
- $\mathbf{u}_h = \mathbf{u}_h(\mathbf{P}(\mathbf{f}))$: data-dependence consistent with iNSE
- optimal order (k+1) (discrete inf-sup)

Non-optimality of Taylor-Hood

$$\mathbf{v}_{\sigma,h} \in \mathbf{L}^2_{\sigma,h} \Rightarrow$$

$$\begin{aligned} (\nabla p, \mathbf{v}_{\sigma,h}) &= (\nabla(p - L_h p), \mathbf{v}_{\sigma,h}) \\ &\leq \|\nabla(p - L_h p)\|_{L^2} \|\mathbf{v}_{\sigma,h}\|_{L^2} \quad \Rightarrow \end{aligned}$$

$$\boxed{\|\mathbb{P}_h(\nabla p)\|_{\mathbf{L}^2} \leq \|\nabla(p - L_h p)\|_{L^2}}$$

Reference: A. Linke, L. Rebholz: Pressure-induced locking [...], JCP, 2019.

consistency error: depends on H^1-approximation in $Q_h \cap H^1$!

Non-optimality of Taylor-Hood

$$\underbrace{\|\mathbf{u}_h(T) - \mathbb{P}_h(\mathbf{u}(T))}_{\text{optimal convergence order k+1}}\|_{\mathbf{L}^2} \leq \int_0^T \|\mathbb{P}_h(\nabla p)\|_{\mathbf{L}^2}\,\mathrm{dt}$$

$$\boxed{\|\mathbb{P}_h(\nabla p)\|_{\mathbf{L}^2} \leq \|\nabla(p - L_h p)\|_{L^2} \leq Ch^{k-1}|p|_k}$$

Reference: A. Linke, L. Rebholz: Pressure-induced locking [...], JCP, 2019.

Taylor-Hood: loss of two convergence orders

velocity accuracy limited by pressure approximation properties in H^1-norm

Non-optimality of non-pressure-robust solvers

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

$$\nu \ll 1$$
$$\sqrt(\nu) \leq h$$

Reference: A. Linke, L. Rebholz: Pressure-induced locking […], JCP, 2019.

What are preasymptotic convergence orders in $\mathbf{L}^2$

Non-optimality of non-pressure-robust solvers

Element	Optimal order (L^2)	Achieved order
Bernardi-Raugel $\mathbf{P}_1$(bubble)-P_0	$O(h^2)$	$O(1)$
Crouzeix-Raviart $\mathbf{P}_1$(nc)-P_0	$O(h^2)$	$O(1)$
Mini element $\mathbf{P}_1$(bubble)-P_1	$O(h^2)$	$O(h)$
Taylor-Hood ($\mathbf{P}_2$-P_1)	$O(h^3)$	$O(h)$
$\mathbb{P}_k - \mathbb{P}_{k-1}$	$O(h^{k+1})$	$O(h^{k-1})$
DG-$\mathbb{BDM}_k - \mathbb{P}^{dc}_{k-1}$	$O(h^{k+1})$	$O(h^{k+1})$

Reference: A. Linke, L. Rebholz: Pressure-induced locking […], JCP, 2019.

- Stokes-inf-sup: higher order pressure space needed !
- optimality: Stokes-inf-sup stability & pressure-robustness
- non-pressure-robust: transient Stokes order $\min(k_{\mathbf{u}}+1, k_p)$

Example 3: Steady Stokes flow

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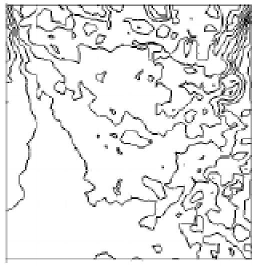
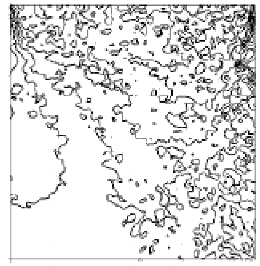
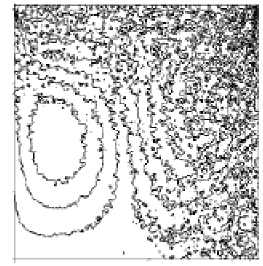
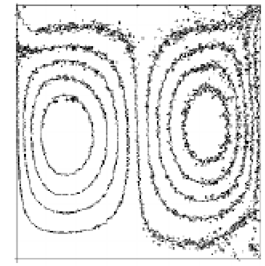
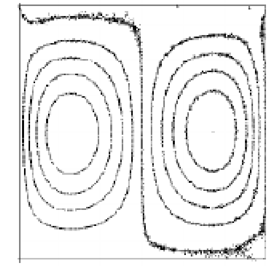
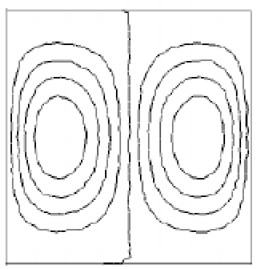
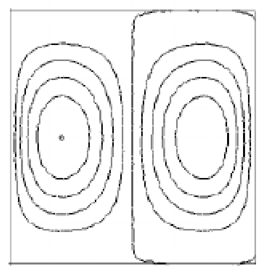
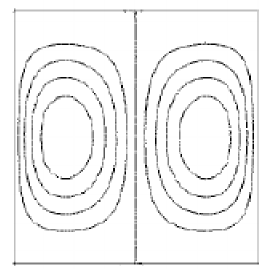
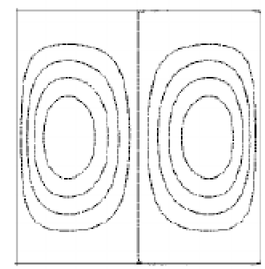
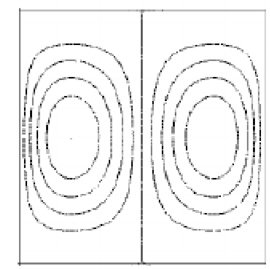
CR					
CR^+					

- new understanding: CR-FEM forcing too strong
- accurate discrete $\mathbf{L}^2$ $-$scalar product (test functions) required

Non-pressure-robust	Pressure-robust

$$-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

inconsistent data dependence = too strong forcing = large errors

$$\mathbf{u}_h = \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right)\left(\frac{1}{\nu}\mathbf{f}\right)$$
$$= \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right)\left(-\Delta \mathbf{u}\right)$$
$$+ \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right)\left(\frac{1}{\nu}\nabla p\right)$$

$$\mathbf{u}_h = \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right)\left(\frac{1}{\nu}\mathbb{P}(\mathbf{f})\right)$$
$$= \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right)\left(\mathbb{P}(-\Delta \mathbf{u})\right)$$

steady Stokes: T-dependence of velocity error replaced by $\frac{1}{\nu}$ -dependence

Improved understanding of divergence constraint

(degenerate) derived evolution equation for divergence

discrete inf-sup

accurate for divergence-free forces

Divergence constraint $-\nabla \cdot \mathbf{u} = g$ involves:

- inf-sup condition
- semi-norm & equivalence-classes of forces

pressure-robustness

derived evolution equation for vorticity

accurate for gradient forces

Last question – the decisive one

How do dominant pressure gradients develop?

Reference:

N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

Model problem 2

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$
$$\nabla \cdot \mathbf{u} = 0$$

model setting

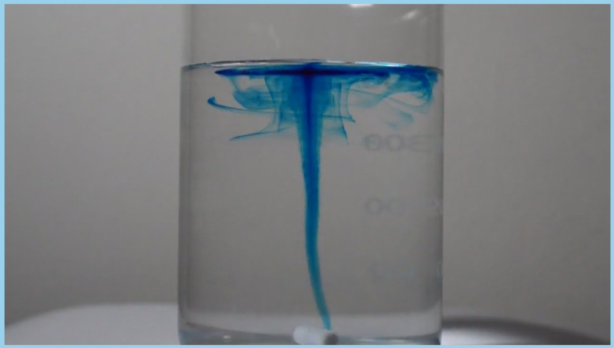
Model problem 2

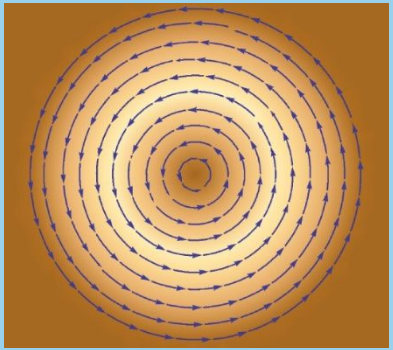
$$\Rightarrow \qquad \frac{D\mathbf{u}}{\mathrm{Dt}} := \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p$$

incompressible Euler flow: material derivative – a gradient field!

Model problem 2 – vortex dominated flows

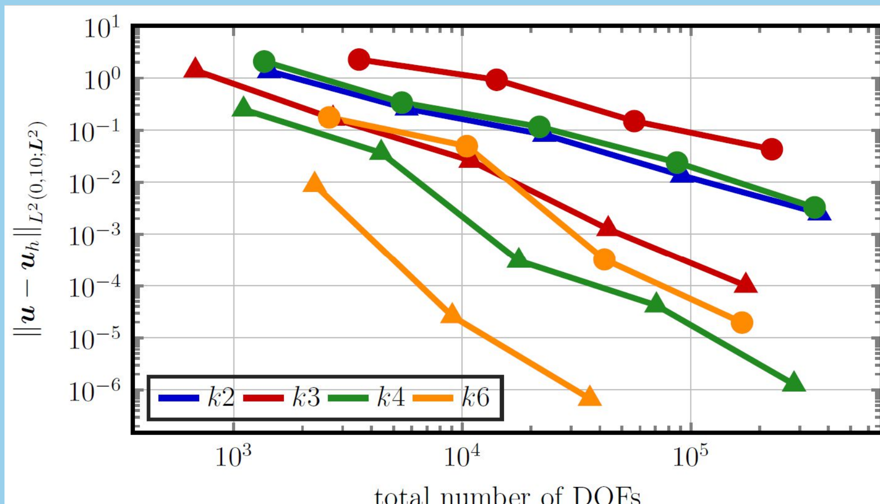
$$\frac{D\mathbf{u}}{\mathrm{Dt}} := \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$

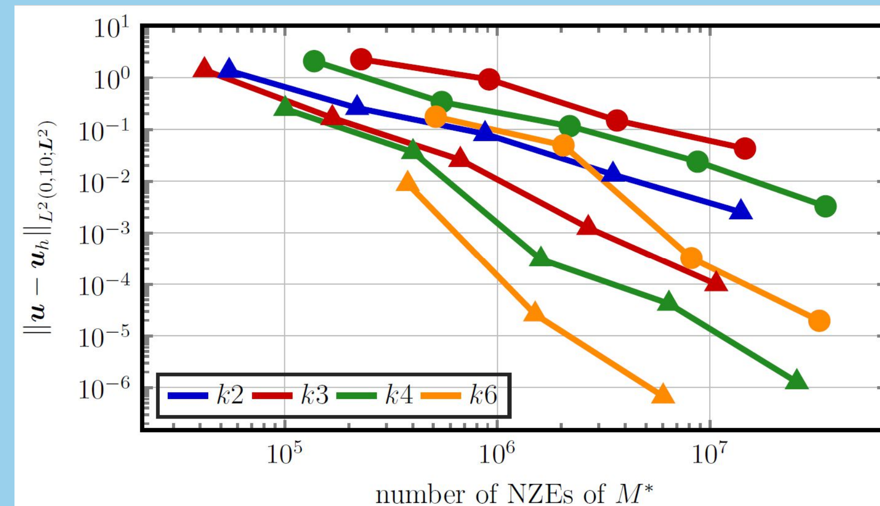




- force balance: centrifugal force = pressure gradient
- quadratic nonlinear convection balances linear pressure gradient
- strong complicated pressure gradient

Example 2: Planar lattice flow





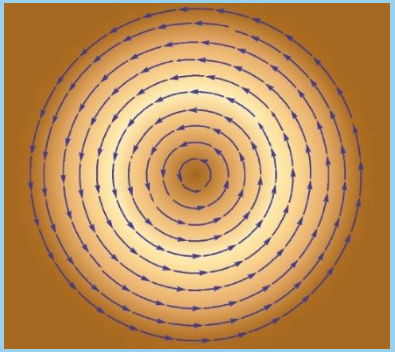
$$\underbrace{\mathbb{BDM}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}}_{\text{pressure-robust}} \quad \text{vs.} \quad \mathbb{P}^{\mathrm{dc}}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}$$

coarse grids: non-pressure-robust solvers lose half of (formal) convergence order

Reference: : N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

Remark on standing Gresho vortex

$$(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 = -\nabla p$$
$$\nabla \cdot \mathbf{u}_0 = 0$$



$$(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 \simeq \mathbf{0}$$

Reference: N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

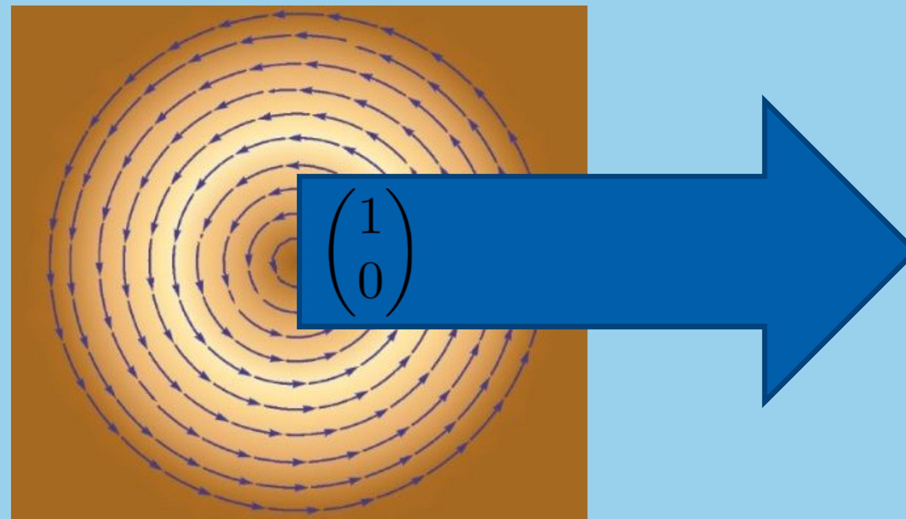
Standing Gresho vortex:

- initial value fulfills steady incompressible Euler equations
- pseudo-dominant convection

Remark on moving Gresho vortex

initial value:

$$\mathbf{u}_0 \to \begin{pmatrix}1\\0\end{pmatrix} + \mathbf{u}_0$$

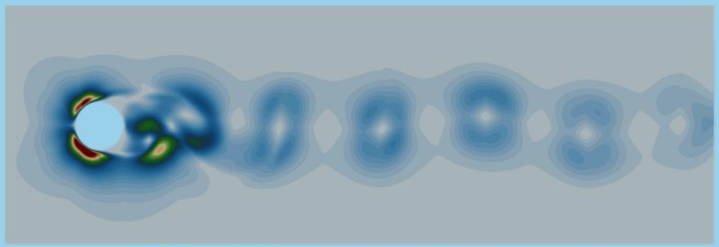


$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 + \partial_x\mathbf{u}_0 \simeq \underbrace{\partial_x\mathbf{u}_0}_{\text{divergence-free}}$$

Reference: N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

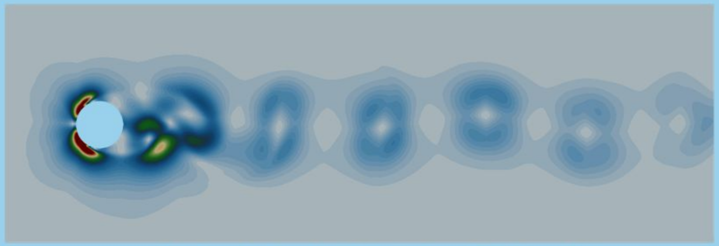
- moving Gresho vortex: Galilean-invariant to standing vortex
- divergence-free forces: motion of vortex (upwind)
- gradient forces: avoid vortex destruction (pressure-robustness)

Karman vortex street Re=100



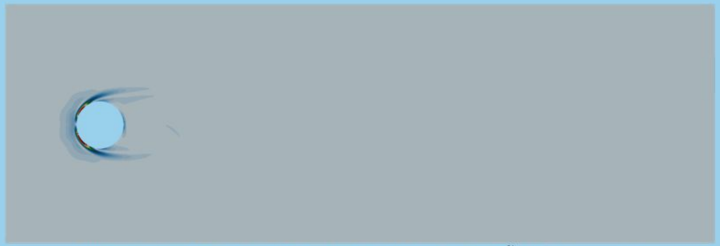
material derivative $|f_h^t = \partial_t u_h + (u_h \cdot \nabla_h) u_h|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50



gradient part of material derivative $|\nabla \phi_h^t|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50



divergence-free part of material derivative $|\mathbb{P}_h^{\mathrm{div}}(f_h^t)|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50

$$\frac{D\mathbf{u}}{\mathrm{Dt}} = -\nabla p + \underbrace{\nu \Delta \mathbf{u}}_{\text{divergence-free}}$$

Reference: N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

- material derivative: small divergence-free part
- pressure-robust schemes: better around obstacle

Classification of pressure-robust solvers

pressure-robustness: H(div)-conforming discretization for incompressible Euler part

- **H**(div)-conforming DG: G. Kanschat, B. Cockburn, D. Schötzau, J. Schöberl, C. Lehrenfeld, (NGSOLVE !!!), C. Cotter, …
- $\mathbf{H}^1$-conforming ´divergence-free´ schemes: Scott-Vogelius, M. Neilan, J. Guzman, A. Buffa, …
- IGA: T. Hughes, J. Evans, …
- conforming & non-conforming schemes with **H**(div)-conforming velocity reconstructions: A. Linke, C. Merdon, L. Tobiska, G. Matthies, A. Ern, D. di Pietro, F. Schieweck, P. Lederer, J. Schöberl, C. Lehrenfeld, W. Wollner, P. Zanotti, C. Kreuzer, R. Verfürth, …

transient Navier-Stokes: Gradient Schemes (J. Droniou, R. Eymard, T. Gallouet, C. Guichard, R. Herbin)

Alternative:

- direct discretization of vorticity equation (in 2d, periodic boundary conditions,)

Messages

Divergence constraint in incompressible flows:

- dominant gradients: source for numerical errors in CFD (just by pressure-dependence of velocity error)
- Stokes-inf-sup & pressure-robustness yield optimality (time-dependent Stokes problem)
- semi-norm: confusion, e.g., pseudo-dominant convection
- CFD: restart out of confusion: possible & necessary

Outlook

- pressure-robust a-posteriori error estimators for pressure-robust schemes: C. Merdon, J. Schöberl, P. Lederer
- pressure-robust convection stabilization
- extensions to low Mach number compressible flows: C. Merdon, A. Linke, T. Gallouet, R. Herbin, M. Akbas
- anisotropic grids