An introduction to Discrete de Rham (DDR) methods

Daniele A. Di Pietro

Institut Montpelliérain Alexander Grothendieck, University of Montpellier

https://imag.umontpellier.fr/~di-pietro/

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https://erc-nemesis.eu/events/workshop-montpellier

Outline

Setting I

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- We have $b_0 = 1$ (number of connected components) and $b_3 = 0$
- b_1 accounts for the number of tunnels crossing Ω

$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

- b_2 is the number of voids encapsulated by Ω

$$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$$

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Setting II

- We consider PDE models that hinge on the vector calculus operators:

$$\mathbf{grad}\, q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \ \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \ \mathrm{div}\, \boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

  for smooth enough functions

$$q : \Omega \to \mathbb{R}, \qquad \boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \boldsymbol{w} : \Omega \to \mathbb{R}^3$$

- The corresponding L^2-graph (domain) spaces are

$$\begin{aligned} H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \ : \ \mathbf{grad}\, q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3 \right\}, \\ \boldsymbol{H}(\mathbf{curl}; \Omega) &:= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\}, \\ \boldsymbol{H}(\mathrm{div}; \Omega) &:= \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega) \right\} \end{aligned}$$

- Assume for the moment that Ω has trivial topology (i.e., $b_1 = b_2 = 0$)

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Three model problems

The Stokes problem in curl-curl formulation

- Given a real number $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads: Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\textstyle\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

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- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$, the magnetostatics problem reads: Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu\boldsymbol{H} - \mathbf{curl}\,\boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div}\boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\operatorname{div};\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall\boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
\int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \operatorname{div}\boldsymbol{A}\operatorname{div}\boldsymbol{v} &= \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} && \forall\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)
\end{aligned}
$$

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- Given $\kappa > 0$ and $f \in L^2(\Omega)$, the Darcy problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\kappa^{-1}\boldsymbol{u} - \mathbf{grad}\, p &= 0 && \text{in } \Omega, && \text{(Darcy's law)} \\
-\operatorname{div} \boldsymbol{u} &= f && \text{in } \Omega, && \text{(mass conservation)} \\
p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \kappa^{-1}\boldsymbol{u} \cdot \boldsymbol{v} + \int_\Omega p \operatorname{div} \boldsymbol{v} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega), \\
-\int_\Omega \operatorname{div} \boldsymbol{u}\, q &= \int_\Omega f\, q && \forall q \in L^2(\Omega)
\end{aligned}
$$

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A unified view

- The above problems are mixed formulations involving two fields
- They can be recast into the abstract setting: Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma,v) + c(u,v) &= g(v) \quad \forall v \in U, \end{aligned}$$

  or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma,u),(\tau,v)) = f(\tau) + g(v) \qquad \forall (\tau,v) \in \Sigma \times U$$

  with

$$\mathcal{A}((\sigma,u),(\tau,v)) \coloneqq a(\sigma,\tau) + b(\tau,u) - b(\sigma,v) + c(u,v) = f(\tau) + g(v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool for well-posedness: The de Rham complex

$$\{0\} \longrightarrow H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties, possibly depending on the topology of Ω:

$$\begin{aligned} \mathrm{Im}\,\mathbf{grad} &\subset \mathrm{Ker}\,\mathbf{curl}, \\ \mathrm{Im}\,\mathbf{curl} &\subset \mathrm{Ker}\,\mathrm{div}, \\ \Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} &= L^2(\Omega) \end{aligned}$$

(Darcy, magnetostatics)

A unified tool for well-posedness: The de Rham complex

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- Key properties, possibly depending on the topology of Ω:

no tunnels crossing Ω $(b_1 = 0) \implies \mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl}$ (Stokes)

no voids contained in Ω $(b_2 = 0) \implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div}$ (magnetostatics)

$\Omega \subset \mathbb{R}^3$ $(b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega)$ (Darcy, magnetostatics)

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$\Omega \subset \mathbb{R}^3$ $(b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega)$ (Darcy, magnetostatics)

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$\mathrm{Ker}\,\textbf{curl}\,/\mathrm{Im}\,\textbf{grad}$ and $\mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\textbf{curl}$

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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\textbf{curl}\,/\mathrm{Im}\,\textbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\textbf{curl}$$

- **Emulating these algebraic properties is key for stable discretizations**

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Generalization through differential forms

- The de Rham complex generalizes to domains of $\mathbb{R}^n$ or smooth manifolds
- Denoting by d the exterior derivative and by $H\Lambda(\Omega)$ its L^2-domain,

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the vector calculus version is recovered through vector proxies

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

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The (trimmed) Finite Element way

Local spaces

- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $k \ge -1$,

$$\mathcal{P}^k(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree} \le k \text{ to } T\}$$

- Fix $k \ge 0$ and write, denoting by $\boldsymbol{x}_T$ a point inside T,

$$\begin{aligned}\mathcal{P}^k(T)^3 &= \mathbf{grad}\,\mathcal{P}^{k+1}(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}^{k-1}(T)^3 \eqqcolon \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) \\ &= \mathbf{curl}\,\mathcal{P}^{k+1}(T)^3 \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}^{k-1}(T) \qquad \eqqcolon \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)\end{aligned}$$

- Define the trimmed spaces that sit between $\mathcal{P}^k(T)^3$ and $\mathcal{P}^{k+1}(T)^3$:

$$\begin{aligned}\boldsymbol{\mathcal{N}}^{k+1}(T) &\coloneqq \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(T) \\ \boldsymbol{\mathcal{RT}}^{k+1}(T) &\coloneqq \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k+1}(T)\end{aligned}$$

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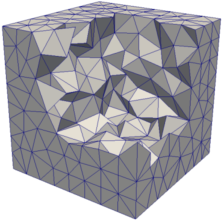
$$\boldsymbol{\mathcal{RT}}^{k+1}(T) \coloneqq \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k+1}(T)$$

- $\mathcal{P}^{-,k}\Lambda^r(f)$ generalizes to r-forms on d-faces f through Koszul complements

The (trimmed) Finite Element way

Global complex



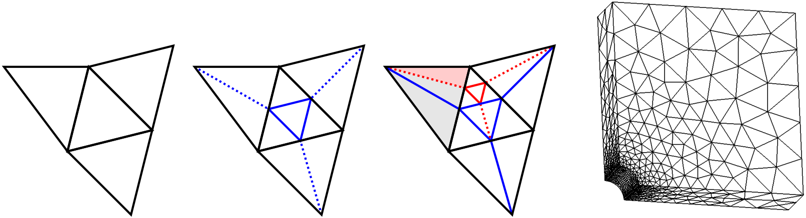
- Let $\mathcal{T}_h$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form a global FE complex:

$$
\begin{array}{ccccccccc}
\mathcal{P}_c^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{N}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^k(\mathcal{T}_h) & \longrightarrow & \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}
$$

- **The gluing only works on conforming meshes (simplicial complexes)!**

The Finite Element way

Shortcomings



- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

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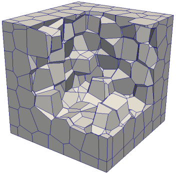
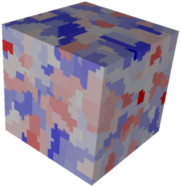
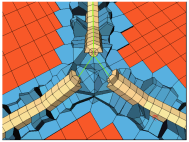
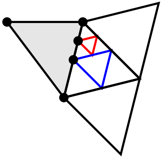
The discrete de Rham (DDR) approach I

- **Key idea:** replace both spaces and operators by discrete counterparts:

$$\underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

- Support of polyhedral meshes (CW complexes) and high-order
- Several strategies to reduce the number of unknowns on general shapes
- Natural generalization to the de Rham complex of differential forms

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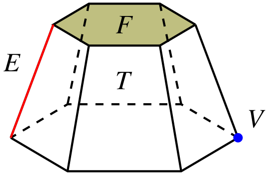
The discrete de Rham (DDR) approach II



- DDR spaces are spanned by vectors of polynomials
- Polynomial components enable consistent reconstructions of
  - Vector calculus operators
  - The corresponding scalar or vector potentials
- These reconstructions emulate integration by parts (Stokes) formulas

References for this presentation

- Vector FE spaces [Raviart and Thomas, 1977, Nédélec, 1980]
- FEEC [Arnold, Falk, Winther, 2006–pres.]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR with Koszul complements [DP and Droniou, 2023a]
- Algebraic properties (general topology) [DP, Droniou, Pitassi, 2023]
- Bridges with VEM [Beirão da Veiga, Dassi, DP, Droniou, 2022]
- Polytopal Exterior Calculus (PEC) [Bonaldi, DP, Droniou, Hu, 2023]
- C++ open-source implementation available in HArDCore3D

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Outline

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- With F mesh face let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q := (\mathbf{grad}_F\, q)^\perp \qquad \mathrm{rot}_F\, \boldsymbol{v} := \mathrm{div}_F(\boldsymbol{v}^\perp)$$

- We start by deriving a discrete counterpart of the 2D de Rham complex:

$$H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \longrightarrow \{0\}$$

- We will need the following decomposition of $\mathcal{P}^k(F)^2$:

$$\mathcal{P}^k(F)^2 = \mathbf{rot}_F\, \mathcal{P}^{k+1}(F) \oplus (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F) =: \mathcal{R}^k(F) \oplus \mathcal{R}^{\mathrm{c},k}(F),$$

and recall the 2D Raviart–Thomas space

$$\mathcal{RT}^{k+1}(F) := \mathcal{R}^k(F) \oplus \mathcal{R}^{\mathrm{c},k+1}(F)$$

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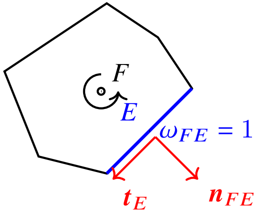

- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

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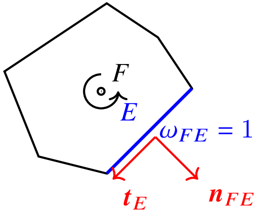

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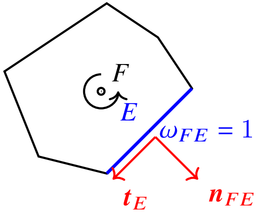
$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

The two-dimensional case

A key remark



- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$

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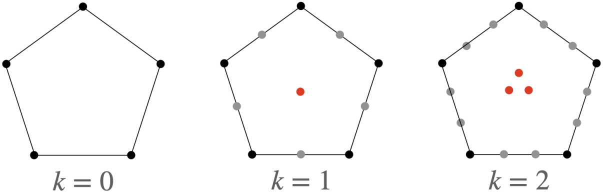
- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\boxed{\underline{X}^k_{\mathbf{grad},F} \coloneqq \left\{\underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\rm c}(\mathcal{E}_F)\right\}}$$

- Let $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ be s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q \coloneqq (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \ \forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \ \forall V \in \mathcal{V}_F$$



- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The face gradient $\mathbf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- The scalar trace $\gamma^{k+1}_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^{k+1}(F)$ is s.t., for all $\boldsymbol{v} \in \mathcal{R}^{\mathrm{c},k+2}(F)$,

$$\boxed{\int_F \gamma^{k+1}_F \underline{q}_F \operatorname{div}_F \boldsymbol{v} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency: For all $q \in \mathcal{P}^{k+1}(F)$,

$$\mathbf{G}^k_F(\underline{I}^k_{\mathbf{grad},F} q) = \mathbf{grad}_F\, q \text{ and } \gamma^{k+1}_F(\underline{I}^k_{\mathbf{grad},F} q) = q$$

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- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{RT}}^{k+1}(F)^{\perp}$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F \boldsymbol{v} \; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F \, q}_{\in \boldsymbol{\mathcal{RT}}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) q_{|E}$$

The two-dimensional case

Discrete $\boldsymbol{H}(\mathrm{rot}; F)$ space

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{RT}}^{k+1}(F)^\perp$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F \boldsymbol{v}\ q = \int_F \boldsymbol{\pi}^k_{\mathcal{RT},F}\boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{RT}}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v} \cdot \boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} q_{|E}$$

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The two-dimensional case

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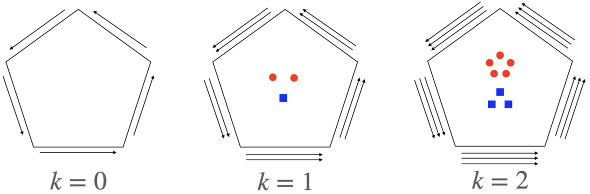
- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$:= $\mathcal{RT}^{k+1}(F)^{\perp}$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\ q = \int_F \boldsymbol{\pi}^k_{\mathcal{RT},F}\boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \mathcal{RT}^k(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v} \cdot \boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} q_{|E}$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot}; F)$:

$$\boxed{\underline{\boldsymbol{X}}^k_{\mathbf{curl},F} := \left\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_F, (v_E)_{E \in \mathcal{E}_F}) \ :\ \boldsymbol{v}_F \in \mathcal{RT}^k(F) \text{ and } v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \right\}}$$

- $\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} : H^1(F)^2 \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ is obtained collecting L^2-orthogonal projections



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- The face curl operator $C^k_F : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C^k_F \underline{\boldsymbol{v}}_F \, q = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \, q \quad \forall q \in \mathcal{P}^k(F)}$$

- The tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$ is s.t.,

$$\boxed{\begin{aligned}\int_F \boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, r + \boldsymbol{w}) &\\ = \int_F C^k_F \underline{\boldsymbol{v}}_F \, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \, r + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w} &\\ \forall (r, \boldsymbol{w}) \in \mathcal{P}^{0,k+1}(F) \times \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F)&\end{aligned}}$$

- We have the following polynomial consistency:

$$C^k_F \left(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v}\right) = \mathrm{rot}_F \, \boldsymbol{v} \ \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) \text{ and } \boldsymbol{\gamma}^k_{\mathrm{t},F} \left(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v}\right) = \boldsymbol{v} \ \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$$

Two-dimensional DDR complex

Space	V (vertex)	E (edge)	F (face)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

- Define the discrete gradient

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F := \left(\boldsymbol{\pi}^k_{\mathcal{RT},F} \mathsf{G}^k_F \underline{q}_F, (G^k_E \underline{q}_E)_{E\in\mathcal{E}_F}\right)$$

- The two-dimensional DDR complex reads

$$\underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \longrightarrow \{0\}$$

- If F is simply connected, this complex is exact

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A glance at the general case I

- $\underline{X}_h^{k,r}$ spanned by vectors of polynomial components
- Recursive and hierarchical construction on d-cells, $d = r+1, \ldots, n$, of
  - A discrete exterior derivative

$$\mathrm{d}_f^{k,r} : \underline{X}_f^{k,r} \to \mathcal{P}^k \Lambda^{r+1}(f)$$

  - Based on it, an associated discrete potential ($\simeq k$-form inside f)

$$P_f^{k,r} : \underline{X}_f^{k,r} \to \mathcal{P}^k \Lambda^r(f)$$

- Reconstructions mimic the Stokes formula: $\forall(\omega, \mu) \in \Lambda^\ell(f) \times \Lambda^{n-\ell-1}(f)$,

$$\int_f \mathrm{d}^\ell \omega \wedge \mu = (-1)^{\ell+1} \int_f \omega \wedge \mathrm{d}^{n-\ell-1} \mu + \int_{\partial f} \mathrm{tr}_{\partial f}\, \omega \wedge \mathrm{tr}_{\partial f}\, \mu$$

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A glance at the general case II

- For a polytopal domain $\Omega \subset \mathbb{R}^n$ and a form degree r, the DDR space is

$$\underline{X}_h^{k,r} := \bigtimes_{d=r}^{n} \bigtimes_{f \in \Delta_d(\mathcal{T}_h)} \mathcal{P}^{-,k}\Lambda^{d-r}(f) \text{ with } \Delta_d(\mathcal{T}_h) := \{d\text{-faces of } \mathcal{T}_h\}$$

- We recursively define, for $f \in \Delta_d(\mathcal{T}_h)$, $d = r, \ldots, n$,
  - If $r = d$,

$$P_f^{k,d} \underline{\omega}_f := \star^{-1} \omega_f \in \mathcal{P}^k \Lambda^d(f)$$

  - If $r + 1 \le d \le n$, we first let, for all $\underline{\omega}_f \in \underline{X}_f^{k,r}$ and all $\mu \in \mathcal{P}^k \Lambda^{d-r-1}(f)$,

$$\int_f \mathrm{d}_f^{k,r} \underline{\omega}_f \wedge \mu = (-1)^{r+1} \int_f \star^{-1} \omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P_{\partial f}^{k,r} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu$$

then, for all $(\mu, \nu) \in \kappa \mathcal{P}^{k,d-r}(f) \times \kappa \mathcal{P}^{k-1,d-r+1}(f)$,

$$\begin{aligned}(-1)^{r+1} \int_f P_f^{k,r} \underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) &= \int_f \mathrm{d}_f^{k,r} \underline{\omega}_f \wedge \mu \\ &\quad - \int_{\partial f} P_{\partial f}^{k,r} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu + (-1)^{r+1} \int_f \star^{-1} \omega_f \wedge \nu\end{aligned}$$

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A glance at the general case III

- The following polynomial consistency properties hold:

$$\begin{aligned} P_f^{k,r} \underline{I}_f^{k,r} \omega &= \omega && \forall \omega \in \mathcal{P}^k \Lambda^r(f), \\ \mathrm{d}_f^{k,r} \underline{I}_f^{k,r} \omega &= \mathrm{d}\omega && \forall \omega \in \mathcal{P}^{-,k+1} \Lambda^r(f) \end{aligned}$$

- Setting

$$\underline{\mathrm{d}}_h^{k,r} \underline{\omega}_h \coloneqq \left(\pi_f^{-,k,d-r-1} (\star \mathrm{d}_f^{k,r} \underline{\omega}_f) \right)_{f \in \Delta_d(\mathcal{T}_h),\, d \in [k+1,n]},$$

the global DDR complex of differential forms reads

$$\underline{X}_h^{k,0} \xrightarrow{\underline{\mathrm{d}}_h^{k,0}} \underline{X}_h^{k,1} \longrightarrow \cdots \longrightarrow \underline{X}_h^{k,n-1} \xrightarrow{\underline{\mathrm{d}}_h^{k,n-1}} \underline{X}_h^{k,n} \longrightarrow \{0\}$$

A glance at the general case IV

For $n = 3$, vector proxies yield the DDR complex of [DP and Droniou, 2023a]:

$$\underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \longrightarrow \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^{k,0}_T \cong \underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}^{k,1}_T \cong \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\boldsymbol{\mathcal{RT}}^k(F)$	$\boldsymbol{\mathcal{RT}}^k(T)$
$\underline{X}^{k,2}_T \cong \underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\underline{X}^{k,3}_T \cong \mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Commutation with the interpolators

Lemma (Local commutation properties)

The following diagrams commute:

$$
\begin{array}{ccccccccc}
C^\infty(\overline{T}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{T}) & \longrightarrow & \{0\} \\
\Big\downarrow {\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{grad},T}} & & \Big\downarrow {\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},T}} & & \Big\downarrow {\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},T}} & & \Big\downarrow {\scriptstyle i_T} & & \\
\underline{X}^k_{\mathbf{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T) & \longrightarrow & \{0\}
\end{array}
$$

- Crucial for both algebraic and analytical properties
- Compatibility of projections with Helmholtz–Hodge decompositions

$\Longrightarrow$ robustness of DDR schemes with respect to the physics, e.g.:

  - Stokes [Beirão da Veiga, Dassi, DP, Droniou, 2022]
  - Navier–Stokes [DP, Droniou, Qian, 2023]
  - Reissner–Mindlin [DP and Droniou, 2023b]

Local discrete L^2-products

- Based on the element potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are built to be polynomially consistent

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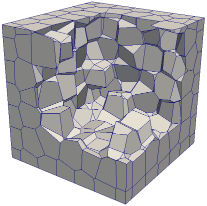
Global DDR complex



$$\underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{G}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

- Global DDR spaces on a mesh $\mathcal{T}_h$ are defined gluing boundary components
- Global operators are obtained collecting local components
- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise

Cohomology of the global three-dimensional DDR complex

$$\underline{\boldsymbol{X}}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow \{0\}$$

Theorem (Cohomology of the 3D DDR complex [DP, Droniou, Pitassi, 2023])

For any $k \geq 0$, the DDR sequence forms a complex whose cohomology spaces are isomorphic to those of the continuous de Rham complex. In particular, if Ω has a trivial topology (i.e., $b_1 = b_2 = 0$), the DDR complex is exact, i.e.,

$$\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h, \quad \mathrm{Im}\,\underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\,D^k_h, \quad \mathrm{Im}\,D^k_h = \mathcal{P}^k(\mathcal{T}_h).$$

Remark (Extension to PEC [Bonaldi, DP, Droniou, Hu, 2023])

The above result extends to the de Rham complex of differential forms.

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Outline

Uniform discrete Poincaré inequality for the curl

Theorem (Poincaré inequality for the curl [DP and Hanot, 2024])

For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, *it holds*

$$\inf_{\underline{\boldsymbol{z}}_h \in \mathrm{Ker}\, \underline{\boldsymbol{C}}^k_h} \|\underline{\boldsymbol{v}}_h - \underline{\boldsymbol{z}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h},$$

with hidden constant only depending on Ω, *mesh regularity, and* k.

This results holds for domains of general topology!

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Adjoint consistency of the discrete curl

Adjoint consistency measures the failure to satisfy a global IBP. For the curl,

$$\int_\Omega \boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{v} - \int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{v} = 0 \text{ if } \boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega$$

Theorem (Adjoint consistency for the curl)

Let $\mathcal{E}_{\mathbf{curl},h} : \left(C^0(\overline{\Omega})^3 \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)\right) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$ *be s.t.*

$$\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h) := (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h)_{\mathrm{div},h} - \int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h.$$

Then, for all $\boldsymbol{w} \in C^0(\overline{\Omega})^3 \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ *s.t.* $\boldsymbol{w} \in H^{k+2}(\mathcal{T}_h)^3$*:* $\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$*,*

$$|\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h)| \lesssim h^{k+1}\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right).$$

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DDR scheme

- Assume $b_2 = 0$. We seek $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{A}\, \mathrm{div}\, \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- With obvious substitutions: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned} (\mu \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\ (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{A}}_h\, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \end{aligned}$$

- If $b_2 \neq 0$, we need to add orthogonality of $\underline{\boldsymbol{A}}_h$ to harmonic forms

$$\underline{\mathfrak{H}}^k_{\mathrm{div},h} := \left\{ \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \,:\, D^k_h \underline{\boldsymbol{w}}_h = 0 \text{ and } (\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} = 0\ \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \right\}$$

Analysis

- Inf-sup stability is proved as in the continuous case for the norm

$$|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h := \left(\|\underline{\boldsymbol{\tau}}_h\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h\|^2_{\mathrm{div},h} + \|\underline{\boldsymbol{v}}_h\|^2_{\mathrm{div},h} + \|D^k_h \underline{\boldsymbol{v}}_h\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

- Crucial points are the isomorphism in cohomology and Poincaré inequality
- Assuming $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$ and $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \times H^{k+2}(\mathcal{T}_h)^3$, it holds

$$|||(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{A})|||_h \lesssim h^{k+1}$$

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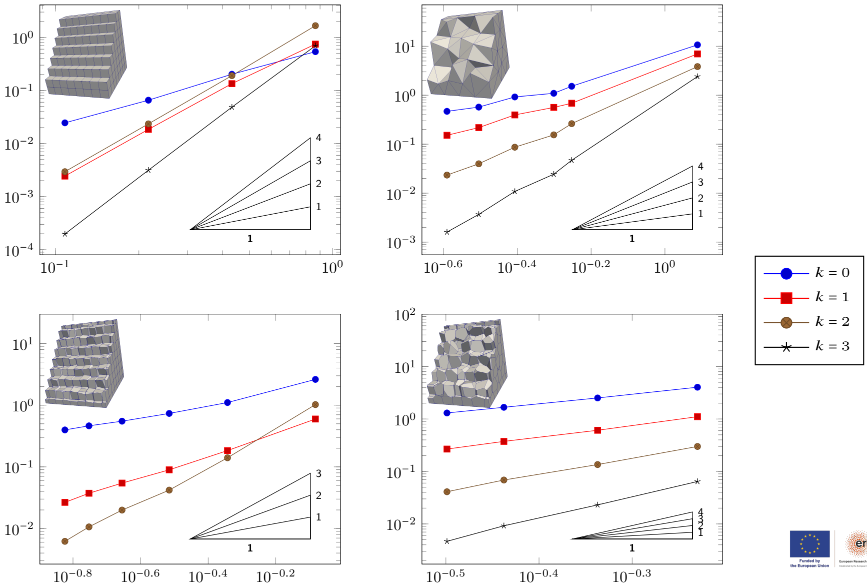
Numerical examples (energy error vs. meshsize)



Conclusions and perspectives

- Fully discrete approach for PDEs relating to the de Rham complex
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to differential forms

- Unified proof of analytical properties using differential forms
- Development of novel complexes (e.g., elasticity, Hessian,. . .)
- Applications (possibly beyond continuum mechanics)

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