A discrete exact grad-curl-div complex on generic polyhedral meshes

Part I: Algebraic properties

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The magnetostatics problem

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedron and $\boldsymbol{f} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}; \Omega)$
- We consider the problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\mathrm{div}; \Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{\sigma} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{u}\, \mathrm{div}\, \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}; \Omega) \end{aligned}$$

- Well-posedness hinges on properties of the de Rham complex

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}; \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Specifically, we need the following exactness properties:

$$\mathrm{Im}\, \mathbf{curl} = \mathrm{Ker}\, \mathrm{div} \text{ if } b_2 = 0, \qquad \mathrm{Im}\, \mathrm{div} = L^2(\Omega)$$

The discrete de Rham (DDR) approach I

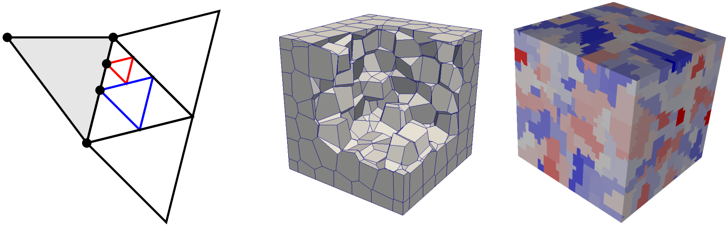


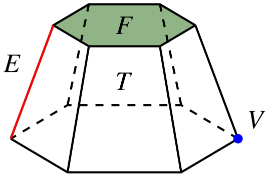
Figure: Examples of polytopal meshes supported by the DDR approach

- **Key idea:** replace spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of general polyhedral meshes and high-order
- Exactness proved at the discrete level (directly usable for stability)

The discrete de Rham (DDR) approach II



- The fully discrete spaces are spanned by vectors of polynomials
- Polynomial components attached to geometric objects
  - emulate the continuity properties of the corresponding space
  - enable the reconstruction of vector calculus operators and potentials
- The key ingredient is the Stokes formula

- Let F be a mesh face and set, for smooth $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$,

$$\mathbf{rot}_F\, q \coloneqq \varrho_{-\pi/2}(\mathbf{grad}_F\, q) \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneqq \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v})$$

- We derive a discrete counterpart of the two-dimensional local complex:

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

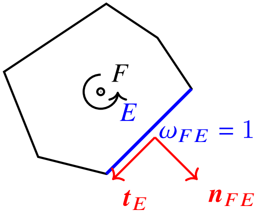
- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

$$\begin{aligned}\mathcal{P}^k(F)^2 &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{G}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp\mathcal{P}^{k-1}(F)}_{\mathcal{G}^{\mathrm{c},k}(F)}\end{aligned}$$

- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\begin{aligned}\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} &= -\int_F q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE}) \\ &= -\int_F \pi_{\mathcal{P},F}^{k-1} q\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})\end{aligned}$$

  with $\pi_{\mathcal{P},F}^{k-1}$ L^2-orthogonal projector on $\mathcal{P}^{k-1}(F)$
- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$



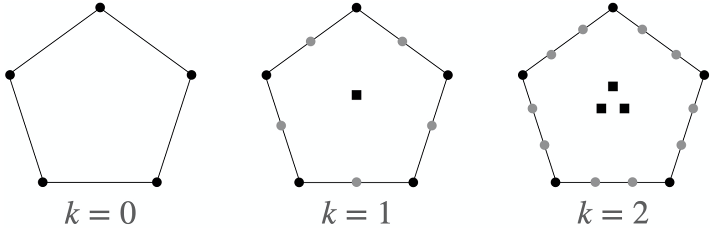


Figure: Number of degrees of freedom for $\underline{X}^k_{\mathbf{grad},F}$ for $k \in \{0, 1, 2\}$

- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\underline{X}^k_{\mathbf{grad},F} := \left\{ \underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_F) \right\}$$

- The interpolator $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ is s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q := (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \ \forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \ \forall V \in \mathcal{V}_F$$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \,\mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F}\, q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- We reconstruct similarly a face potential (scalar trace) in $\mathcal{P}^{k+1}(F)$

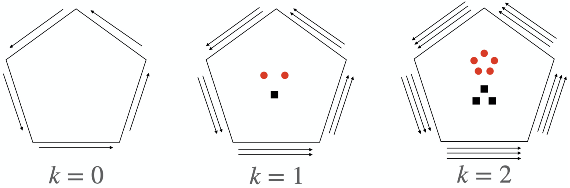


Figure: Number of degrees of freedom for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ for $k \in \{0, 1, 2\}$

- We reason starting from: $\forall \boldsymbol{v} \in \boldsymbol{N}^{k+1}(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) \underbrace{q_{|E}}_{\in \mathcal{P}^k(E)} \qquad \forall q \in \mathcal{P}^k(F)$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot}; F)$:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}, (v_E)_{E \in \mathcal{E}_F}\big) : \boldsymbol{v}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} \to \mathcal{P}^k(F)$ is s.t.,

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)$$

- Define the interpolator $\underline{\boldsymbol{I}}_{\mathbf{curl},F}^{k} : H^1(F)^2 \longrightarrow \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k}$ s.t., $\forall \boldsymbol{v} \in H^1(F)^2$,

$$\underline{\boldsymbol{I}}_{\mathbf{curl},F}^{k} \boldsymbol{v} \coloneqq \left(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1} \boldsymbol{v}, \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k} \boldsymbol{v}, \left(\pi_{\mathcal{P},E}^{k} (\boldsymbol{v}_{|E} \cdot \boldsymbol{t}_E) \right)_{E \in \mathcal{E}_F} \right).$$

- C_F^k is polynomially consistent by construction:

$$C_F^k \left(\underline{\boldsymbol{I}}_{\mathbf{curl},F}^{k} \boldsymbol{v} \right) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- We reconstruct similarly a vector potential (tangent trace) in $\mathcal{P}^k(F)^2$

Theorem (Exactness of the two-dimensional local DDR complex)

If F is simply connected, the following local complex is exact:

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},F}^k} \underline{X}_{\mathbf{grad},F}^k \xrightarrow{\underline{\boldsymbol{G}}_F^k} \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \xrightarrow{C_F^k} \mathcal{P}^k(F) \xrightarrow{0} \{0\},$$

where $\underline{\boldsymbol{G}}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$ *is the discrete gradient s.t.,* $\forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k$,

$$\underline{\boldsymbol{G}}_F^k \underline{q}_F \coloneqq \left(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathsf{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathsf{G}_F^k \underline{q}_F), (G_E^k \underline{q}_F)_{E \in \mathcal{E}_F} \right)$$

The two-dimensional case
Summary

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (face)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

Table: Polynomial components for the two-dimensional spaces

- Interpolators $=$ component-wise L^2-projections
- Discrete operators $=$ L^2-projections of full operator reconstructions

The three-dimensional case I

Exact complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Table: Polynomial components for the three-dimensional spaces

Theorem (Exactness of the three-dimensional local DDR complex)

If the mesh element T has a trivial topology, this complex is exact.

The three-dimensional case

Local discrete L^2-products

- Emulating integration by part formulas, define the local potentials

$$\begin{aligned} P^{k+1}_{\mathbf{grad},T} &: \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T), \\ \boldsymbol{P}^k_{\mathbf{curl},T} &: \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3, \\ \boldsymbol{P}^k_{\mathrm{div},T} &: \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \mathcal{P}^k(T)^3 \end{aligned}$$

- Based on these potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are polynomially exact

- Let $\mathcal{T}_h$ be a polyhedral mesh with elements and faces of trivial topology
- Global DDR spaces are defined gluing boundary components:

$$\underline{X}_{\mathbf{grad},h}^k, \qquad \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k, \qquad \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}_h^k : \underline{X}_{\mathbf{grad},h}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k, \quad \underline{\boldsymbol{C}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \to \underline{\boldsymbol{X}}_{\mathrm{div},h}^k, \quad D_h^k : \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \to \mathcal{P}^k(\mathcal{T}_h)$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise
- The global DDR complex (exact if Ω has a trivial topology) is

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},h}^k} \underline{X}_{\mathbf{grad},h}^k \xrightarrow{\underline{\boldsymbol{G}}_h^k} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \xrightarrow{\underline{\boldsymbol{C}}_h^k} \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Discrete problem

- Continuous problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned} (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\ (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h\, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \end{aligned}$$

- An inf-sup condition follows as in the continuous case leveraging

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- **See J. Droniou's talk for convergence**

References

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