Fully discrete polynomial de Rham complexes on polyhedral meshes wih application to magnetostatics

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The magnetostatics problem

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedron and $\boldsymbol{f} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$
- We consider the problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- Well-posedness hinges on properties of the de Rham complex

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Specifically, we need the following exactness properties:

$$\mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \text{ if } b_2 = 0, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega)$$

The discrete de Rham (DDR) approach I

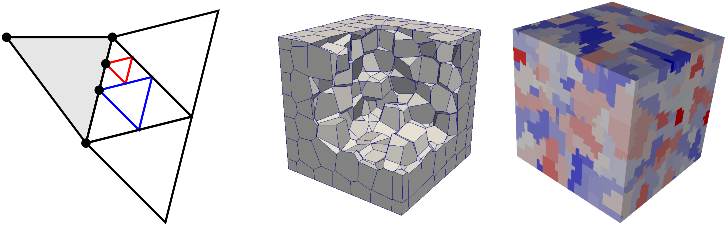


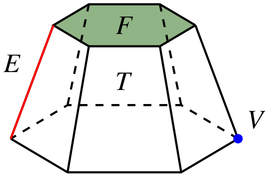
Figure: Examples of polytopal meshes supported by the DDR approach

- **Key idea:** replace spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of general polyhedral meshes and high-order
- Exactness proved at the discrete level (directly usable for stability)

The discrete de Rham (DDR) approach II



- The fully discrete spaces are spanned by vectors of polynomials
- Polynomial components attached to geometric objects
  - emulate the continuity properties of the corresponding space
  - enable the reconstruction of vector calculus operators and potentials
- The key ingredient is the Stokes formula

The two-dimensional case
Continuous exact complex

- Let F be a mesh face and set, for smooth $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$,

$$\mathbf{rot}_F\, q \coloneqq \varrho_{-\pi/2}(\mathbf{grad}_F\, q) \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneqq \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v})$$

- We derive a discrete counterpart of the two-dimensional local complex:

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

$$\begin{aligned}\mathcal{P}^k(F)^2 &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{G}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp\mathcal{P}^{k-1}(F)}_{\mathcal{G}^{\mathrm{c},k}(F)}\end{aligned}$$

The two-dimensional case

A key remark

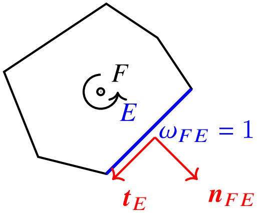
- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

$$= -\int_F \pi_{\mathcal{P},F}^{k-1} q \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

with $\pi_{\mathcal{P},F}^{k-1}$ L^2-orthogonal projector on $\mathcal{P}^{k-1}(F)$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$



The two-dimensional case

Discrete $H^1(F)$ space

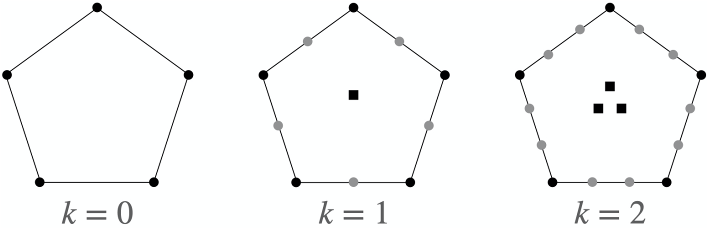


Figure: Number of degrees of freedom for $\underline{X}^k_{\mathbf{grad},F}$ for $k \in \{0,1,2\}$

- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\underline{X}^k_{\mathbf{grad},F} := \left\{ \underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\rm c}(\mathcal{E}_F) \right\}$$

- The interpolator $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ is s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q := (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \;\forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \;\forall V \in \mathcal{V}_F$$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \,\mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- We reconstruct similarly a face potential (scalar trace) in $\mathcal{P}^{k+1}(F)$

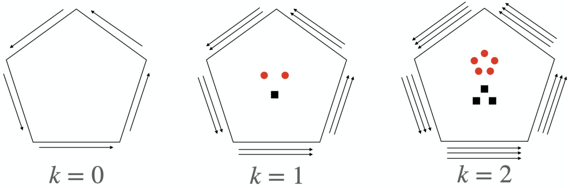


Figure: Number of degrees of freedom for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ for $k \in \{0,1,2\}$

- We reason starting from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) \underbrace{q_{|E}}_{\in \mathcal{P}^k(E)} \qquad \forall q \in \mathcal{P}^k(F)$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot};F)$:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}^{\mathrm{c}}_{\boldsymbol{\mathcal{R}},F}, (v_E)_{E \in \mathcal{E}_F}\big) : \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}^{\mathrm{c}}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)$$

- Define the interpolator $\underline{\boldsymbol{I}}^k_{\mathbf{curl},F} : H^1(F)^2 \longrightarrow \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ s.t., $\forall \boldsymbol{v} \in H^1(F)^2$,

$$\underline{\boldsymbol{I}}^k_{\mathbf{curl},F} \boldsymbol{v} \coloneqq \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F} \boldsymbol{v}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F} \boldsymbol{v}, \left(\pi^k_{\mathcal{P},E} (\boldsymbol{v}_{|E} \cdot \boldsymbol{t}_E) \right)_{E \in \mathcal{E}_F} \right).$$

- C_F^k is polynomially consistent by construction:

$$C_F^k (\underline{\boldsymbol{I}}^k_{\mathbf{curl},F} \boldsymbol{v}) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- We reconstruct similarly a vector potential (tangent trace) in $\mathcal{P}^k(F)^2$

Theorem (Exactness of the two-dimensional local DDR complex)

If F is simply connected, the following local complex is exact:

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathrm{grad},F}^{k}} \underline{X}_{\mathbf{grad},F}^{k} \xrightarrow{\underline{\boldsymbol{G}}_{F}^{k}} \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} \xrightarrow{C_{F}^{k}} \mathcal{P}^{k}(F) \xrightarrow{0} \{0\},$$

where $\underline{\boldsymbol{G}}_{F}^{k} : \underline{X}_{\mathbf{grad},F}^{k} \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k}$ *is the discrete gradient s.t.,* $\forall \underline{q}_{F} \in \underline{X}_{\mathbf{grad},F}^{k}$,

$$\underline{\boldsymbol{G}}_{F}^{k} \underline{q}_{F} := \left(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathsf{G}_{F}^{k} \underline{q}_{F}), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathsf{G}_{F}^{k} \underline{q}_{F}), (G_{E}^{k} \underline{q}_{F})_{E \in \mathcal{E}_{F}} \right)$$

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (face)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

Table: Polynomial components for the two-dimensional spaces

- Interpolators = component-wise L^2-projections
- Discrete operators = L^2-projections of full operator reconstructions

The three-dimensional case I

Exact complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Table: Polynomial components for the three-dimensional spaces

Theorem (Exactness of the three-dimensional local DDR complex)

If the mesh element T has a trivial topology, this complex is exact.

- Emulating integration by part formulas, define the local potentials

$$\begin{aligned} P^{k+1}_{\mathbf{grad},T} &: \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T), \\ \boldsymbol{P}^k_{\mathbf{curl},T} &: \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3, \\ \boldsymbol{P}^k_{\mathrm{div},T} &: \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \mathcal{P}^k(T)^3 \end{aligned}$$

- Based on these potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are polynomially exact

The three-dimensional case

Global complex

- Let $\mathcal{T}_h$ be a polyhedral mesh with elements and faces of trivial topology
- Global DDR spaces are defined gluing boundary components:

$$\underline{X}^k_{\mathbf{grad},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}^k_h : \underline{X}^k_{\mathbf{grad},h} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \quad \underline{\boldsymbol{C}}^k_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}, \quad D^k_h : \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathcal{P}^k(\mathcal{T}_h)$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise
- The global DDR complex is

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Discrete problem

- Continuous problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall\boldsymbol{\tau}\in\boldsymbol{H}(\mathbf{curl};\Omega),\\ \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega)\end{aligned}$$

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned} (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall\underline{\boldsymbol{\tau}}_h\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\ (\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h\underline{\boldsymbol{u}}_h\, D^k_h\underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\mathrm{div},h}\end{aligned}$$

- Stability follows as in the continuous case using exactness properties of

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Convergence

Theorem (Error estimate)

Assume $b_1 = b_2 = 0$, $\boldsymbol{\sigma} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$, $\boldsymbol{u} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$, and set

$$(\underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{e}}_h) := (\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}).$$

Then, we have the following error estimate:

$$\|(\underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{e}}_h)\|_h \le C h^{k+1}\Big(|\,\mathbf{curl}\,\boldsymbol{\sigma}|_{H^{k+1}(\mathcal{T}_h)^3} + |\boldsymbol{\sigma}|_{H^{(k+1,2)}(\mathcal{T}_h)^3} + |\boldsymbol{u}|_{H^{k+1}(\mathcal{T}_h)^3} + |\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^3}\Big),$$

with $\|\cdot\|_h$ discrete (graph) $\boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ norm and C depending only on Ω, k, and mesh regularity.

Numerical examples

Convergence in the energy norm

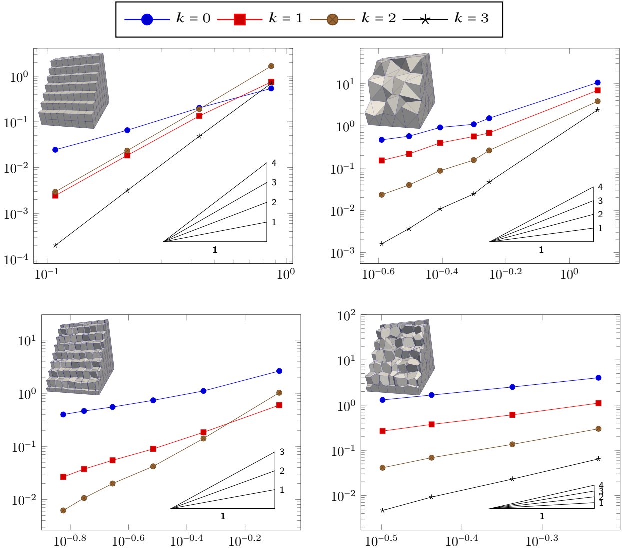


Figure: Energy error versus mesh size h

References I

Di Pietro, D. A. and Droniou, J. (2021a).
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, poincaré inequalities, and consistency.

Di Pietro, D. A. and Droniou, J. (2021b).
An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence.
J. Comput. Phys., 429(109991).

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.