Polytopal Exterior Calculus

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from joint works with F. Bonaldi, J. Droniou, M. Hanot, K. Hu, and S. Pitassi

NEMESIS

New generation methods
for numerical simulations

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References for this presentation

- FEEC [Arnold, Falk, Winther, 2006] and [Arnold, 2018]
- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- DDR with Koszul complements [DP and Droniou, 2023]
- Virtual Elements [Beirão da Veiga, Brezzi, Dassi, Marini, Russo, 2018]
- Polytopal Exterior Calculus (PEC) [Bonaldi, DP, Droniou, Hu, 2024]
- Poincaré (vector calculus formulation) [DP and Hanot, 2024]
- Poincaré (PEC) [DP, Droniou, Hanot, Pitassi, in preparation]

Outline

Outline

1 Preliminaries

2 The Discrete de Rham construction

3 Application to magnetostatics

Setting I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain with Betti numbers b_i
- We have $b_0 = 1$ (number of connected components) and $b_3 = 0$
- b_1 accounts for the number of tunnels crossing Ω

$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

- b_2 is the number of voids encapsulated by Ω

$$(b_0, b_1, b_2, b_3) = (1, 1, 1, 0)$$

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Setting II

- Important PDE models that hinge on the vector calculus operators:

$$\mathbf{grad}\, q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \quad \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \quad \operatorname{div} \boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

  for smooth enough functions

$$q : \Omega \to \mathbb{R}, \qquad \boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \boldsymbol{w} : \Omega \to \mathbb{R}^3$$

- The corresponding L^2-domain spaces are

$$\begin{aligned} H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \ : \ \mathbf{grad}\, q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3 \right\}, \\ \boldsymbol{H}(\mathbf{curl}; \Omega) &:= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\}, \\ \boldsymbol{H}(\operatorname{div}; \Omega) &:= \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \ : \ \operatorname{div} \boldsymbol{w} \in L^2(\Omega) \right\} \end{aligned}$$

The de Rham complex

$$H^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl};\Omega) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\begin{aligned} \mathrm{Im}\,\textbf{grad} &\subset \mathrm{Ker}\,\textbf{curl} \\ \mathrm{Im}\,\textbf{curl} &\subset \mathrm{Ker}\,\mathrm{div} \\ \Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} &= L^2(\Omega) \quad \text{(Darcy, magnetostatics)} \end{aligned}$$

The de Rham complex

$$H^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl};\Omega) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

no tunnels crossing Ω ($b_1 = 0$) $\implies$ $\mathrm{Im}\,\textbf{grad} = \mathrm{Ker}\,\textbf{curl}$ (Stokes)

no voids contained in Ω ($b_2 = 0$) $\implies$ $\mathrm{Im}\,\textbf{curl} = \mathrm{Ker}\,\mathrm{div}$ (magnetostatics)

$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\implies$ $\mathrm{Im}\,\mathrm{div} = L^2(\Omega)$ (Darcy, magnetostatics)

The de Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

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$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\implies$ $\mathrm{Im}\,\mathrm{div} = L^2(\Omega)$ (Darcy, magnetostatics)

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}\,/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\mathbf{curl}$$

The de Rham complex

$$H^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl};\Omega) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\text{no tunnels crossing } \Omega\ (b_1 = 0) \implies \mathrm{Im}\,\textbf{grad} = \mathrm{Ker}\,\textbf{curl} \quad \text{(Stokes)}$$
$$\text{no voids contained in } \Omega\ (b_2 = 0) \implies \mathrm{Im}\,\textbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$
$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\textbf{curl}\,/\mathrm{Im}\,\textbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}\,/\mathrm{Im}\,\textbf{curl}$$

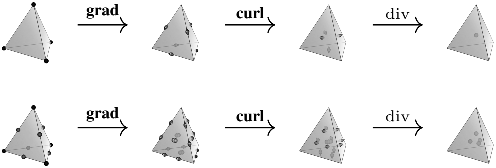
- **Emulating these properties is key for stable discretizations**

The Finite Element way

- Trimmed FE complexes[1] on a tetrahedron T: For any $r \geq 1$,

$r = 1$

$r = 2$



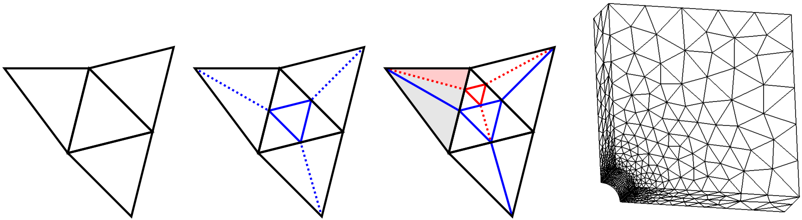
$\dots$

- On a conforming tetrahedral meshes $\mathcal{T}_h$, these spaces can be glued together

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\textbf{grad}} & H(\textbf{curl};\Omega) & \xrightarrow{\textbf{curl}} & H(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{P}_{r,\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\textbf{grad}} & \boldsymbol{\mathcal{N}}_r(\mathcal{T}_h) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{RT}}_r(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{r-1}(\mathcal{T}_h)
\end{array}$$

[1][Raviart and Thomas, 1977, Nédélec, 1980]

Shortcomings



- Approach limited to conforming meshes with standard elements

  ⟹ Local refinement requires to trade mesh size for mesh quality

  ⟹ Complex geometries may require a large number of elements

  ⟹ The element shape cannot be adapted to the solution

- Need for (global) basis functions

  ⟹ Significant increase of DOFs on hexahedral elements

  ⟹ Difficult extension to advanced complexes

A higher-level view of vector calculus operators

- So far, we have treated $\mathbf{grad}$, $\mathbf{curl}$, and div as different operators
- A unified view is possible through exterior calculus
- This view can be exploited, e.g.:
  - To devise unified analysis results for all space dimensions and operators (consistency, Poincaré inequalities, etc.)
  - To construct advanced complexes
  - To work on manifolds (see **J. Droniou**'s presentation)
  - …

Alternating forms

- Let $\{\boldsymbol{e}_i\}_{1\le i\le n}$ denote the canonical basis of $\mathbb{R}^n$ and $\{\mathrm{d}x^i\}_{1\le i\le n}$ its dual s.t.

$$\mathrm{d}x^i(\boldsymbol{e}_j) = \delta_{ij} \qquad 1 \le i, j \le n$$

- An alternating k-form $\omega \in \mathrm{Alt}^k(\mathbb{R}^n)$ is, denoting by $\wedge$ the exterior product,

$$\omega = \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} a_\sigma \,\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}, \quad a_\sigma \in \mathbb{R},$$

- The scalar product in $\mathbb{R}^n$ induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Alt}^k(\mathbb{R}^n)$

Example (Exterior product of 1-forms)

Given $\omega, \mu \in \mathrm{Alt}^1(\mathbb{R}^n)$, $\omega \wedge \mu \in \mathrm{Alt}^2(\mathbb{R}^n)$ is s.t., for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$,

$$(\omega \wedge \mu)(\boldsymbol{v}, \boldsymbol{w}) = \omega(\boldsymbol{v})\mu(\boldsymbol{w}) - \omega(\boldsymbol{w})\mu(\boldsymbol{v}).$$

Notice, in particular, that $\omega \wedge \omega = 0$.

Hodge star I

- The Hodge star operator $\star : \mathrm{Alt}^\ell(\mathbb{R}^n) \to \mathrm{Alt}^{n-\ell}(\mathbb{R}^n)$ is s.t.

$$\forall \omega \in \mathrm{Alt}^\ell(\mathbb{R}^n), \quad \langle \star\omega, \mu \rangle \mathrm{vol} = \omega \wedge \mu \quad \forall \mu \in \mathrm{Alt}^{n-\ell}(\mathbb{R}^n)$$

  where $\mathrm{vol} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the volume form
- It can be checked that $\star$ is an isomorphism
- In what follows, we will also need its inverse

$$\star^{-1} := (-1)^{\ell(n-\ell)} \star$$

Hodge star II

Example (Hodge star)

$n = 2$	$n = 3$
$\star 1 = dx^1 \wedge dx^2$	$\star 1 = dx^1 \wedge dx^2 \wedge dx^3$
$\star dx^1 = dx^2$	$\star dx^1 = dx^2 \wedge dx^3$
$\star dx^2 = -dx^1$	$\star dx^2 = -dx^1 \wedge dx^3$
	$\star dx^3 = dx^1 \wedge dx^2$

Formulas for $\star$ applied to 2- and 3-forms (if $n = 3$) can be obtained taking the $\star^{-1}$ of the previous expressions, e.g.,

$$dx^1 = \star^{-1} \star dx^1 = \star^{-1}(dx^2 \wedge dx^3) = (-1)^{2(3-2)} \star (dx^2 \wedge dx^3) = \star(dx^2 \wedge dx^3).$$

Vector proxies in dimension $n = 3$

For $n = 3$, we can identify vector proxies for all form degrees:

- $\mathrm{Alt}^0(\mathbb{R}^3) := \mathbb{R}$ by definition
- $\mathrm{Alt}^3(\mathbb{R}^3) = \star\mathrm{Alt}^0(\mathbb{R}^3) \cong \mathbb{R}$ since $\star$ is an isomorphism
- $\mathrm{Alt}^1(\mathbb{R}^3) = (\mathbb{R}^3)'$ and, for all $\omega \in \mathrm{Alt}^1(\mathbb{R}^3)$,

$$\omega = a\,\mathrm{d}x^1 + b\,\mathrm{d}x^2 + c\,\mathrm{d}x^3 \cong (a, b, c) \in \mathbb{R}^3$$

- $\mathrm{Alt}^2(\mathbb{R}^3) = \star\mathrm{Alt}^1(\mathbb{R}^3) \cong \mathbb{R}^n$ and, for all $\omega \in \mathrm{Alt}^2(\mathbb{R}^3)$,

$$\omega = a\,\underbrace{\mathrm{d}x^2 \wedge \mathrm{d}x^3}_{\star\mathrm{d}x^1} - b\,\underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^3}_{-\star\mathrm{d}x^2} + c\,\underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^2}_{\star\mathrm{d}x^3} \cong (a, b, c) \in \mathbb{R}^3$$

Differential forms I

- With M open set in an affine subspace of $\mathbb{R}^n$, a (differential) k-form on M is

$$\omega = \sum_{1 \leq \sigma_1 < \cdots < \sigma_k \leq n} a_\sigma \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}, \quad a_\sigma : M \to \mathbb{R}$$

- The value of a k-form at $\boldsymbol{x} \in M$ is denoted $\omega_{\boldsymbol{x}}$:

$$\omega_{\boldsymbol{x}} = \sum_{1 \leq \sigma_1 < \cdots < \sigma_k \leq n} a_\sigma(\boldsymbol{x}) \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k} \in \mathrm{Alt}^k(\mathbb{R}^n)$$

- **A k-form can be integrated on a k-dimensional manifold**

Differential forms II

- The space of k-forms (without regularity requirements on a_σ) is $\Lambda^k(M)$
- When regularity on the a_σ is required, we prepend it to $\Lambda^k(M)$, e.g.,

$L^2\Lambda^k(M)$ = space of k-forms with coefficients a_σ square-integrable on M,

$\mathcal{P}_r\Lambda^k(M)$ = space of k-forms with coefficients a_σ in $\mathcal{P}_r(M)$

Exterior derivative I

- The exterior derivative is the (unbounded) graded operator s.t.

$$\mathrm{d} : L^2\Lambda^k(M) \to L^2\Lambda^{k+1}(M)$$

$$\omega \mapsto \sum_{1\leq\sigma_1<\cdots<\sigma_k\leq n} \sum_{i=1}^{n} \frac{\partial a_\sigma}{\partial x_i} \mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

  which satisfies $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$

- In what follows, we define the domain of the exterior derivative

$$H\Lambda^k(M) := \left\{\omega \in L^2\Lambda^k(M) \;:\; \mathrm{d}\omega \in L^2\Lambda^{k+1}(M)\right\}$$

Exterior derivative II

Example (Exterior derivative of a 1-form)

For a 1-form $C^1\Lambda^1(\overline{\Omega}) \ni \omega = a_1 \mathrm{d}x^1 + a_2 \mathrm{d}x^2 + a_3 \mathrm{d}x^3 \cong \boldsymbol{v}$, we have

$$\begin{aligned}
\mathrm{d}\omega &= \frac{\partial a_1}{\partial x_1}\cancel{\mathrm{d}x^1 \wedge \mathrm{d}x^1} + \frac{\partial a_1}{\partial x_2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial a_1}{\partial x_3}\mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&\quad + \frac{\partial a_2}{\partial x_1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial a_2}{\partial x_2}\cancel{\mathrm{d}x^2 \wedge \mathrm{d}x^2} + \frac{\partial a_2}{\partial x_3}\mathrm{d}x^3 \wedge \mathrm{d}x^2 \\
&\quad + \frac{\partial a_3}{\partial x_1}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial a_3}{\partial x_2}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial a_3}{\partial x_3}\cancel{\mathrm{d}x^3 \wedge \mathrm{d}x^3} \\
&= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\cong \mathbf{curl}\, \boldsymbol{v}.
\end{aligned}$$

The continuous de Rham complex

- In what follows, we will focus on the de Rham complex for a domain Ω of $\mathbb{R}^n$

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, we have the following interpretation in terms of vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

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Outline

General ideas

- Discrete spaces with polynomial components attached to mesh entities
- For any form degree k, recursively on d-cells f, $d = k, \ldots, n$, construct
  - A local discrete potential

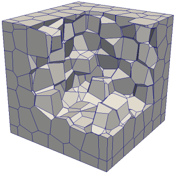
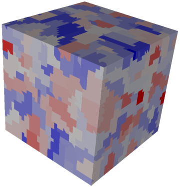
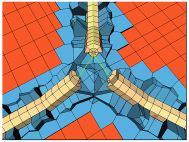
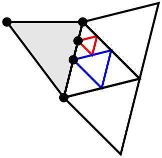
$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^k(f)$$

  - If $d \geq k + 1$, a local discrete exterior derivative

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f)$$

- Connect the spaces through a global discrete exterior derivative

Domain and polytopal mesh

- Assume $\Omega \subset \mathbb{R}^n$ polytopal (polygon if $n = 2$, polyhedron if $n = 3, \ldots$)
- We consider a polytopal mesh $\mathcal{M}_h$ containing all (flat) d-cells, $0 \leq d \leq n$
- d-cells in $\mathcal{M}_h$ are collected in $\Delta_d(\mathcal{M}_h)$, so that, when $n = 3$,
  - $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$ is the set of vertices
  - $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$ is the set of edges
  - $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$ is the set of faces
  - $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$ is the set of elements

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Local Koszul differential and complements I

- Let $f \in \Delta_d(\mathcal{M}_h)$, $d \in [0, n]$, and fix $\boldsymbol{x}_f \in f$
- The Koszul differential $\kappa : \Lambda^{\ell+1}(f) \to \Lambda^{\ell}(f)$ binds the first vector to $\boldsymbol{x} - \boldsymbol{x}_f$:

$$(\kappa\omega)_{\boldsymbol{x}}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell) = \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell)$$

  for all $\boldsymbol{x} \in f$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$ tangent vectors to f
- We define the Koszul complement space

$$\mathcal{K}_r^{\ell}(f) := \kappa \mathcal{P}_{r-1}\Lambda^{\ell+1}(f)$$

Local Koszul differential and complements II

Example (Vector proxies for $\mathcal{K}_r^{\ell}(f_d)$)

ℓ \ d	0	1	2	3
0	$\{0\}$	$\mathcal{P}_r^{\flat}(f_1)$	$\mathcal{P}_r^{\flat}(f_2)$	$\mathcal{P}_r^{\flat}(f_3)$
1		$\{0\}$	$\mathcal{R}_r^{\mathrm{c}}(f_2)$	$\mathcal{G}_r^{\mathrm{c}}(f_3)$
2			$\{0\}$	$\mathcal{R}_r^{\mathrm{c}}(f_3)$
3				$\{0\}$

$$
\begin{aligned}
\mathcal{K}_r^0(f_d) &\cong \mathcal{P}_r^{\flat}(f_d) \coloneqq (\boldsymbol{x}-\boldsymbol{x}_{f_d})\cdot \boldsymbol{\mathcal{P}}_{r-1}(f_d) && \forall d \in \{1,2,3\},\\
\mathcal{K}_r^{d-1}(f_d) &\cong \mathcal{R}_r^{\mathrm{c}}(f_d) \coloneqq (\boldsymbol{x}-\boldsymbol{x}_{f_d})\mathcal{P}_{r-1}(f_d) && \forall d \in \{2,3\},\\
\mathcal{K}_r^1(f_3) &\cong \mathcal{G}_r^{\mathrm{c}}(f_3) \coloneqq (\boldsymbol{x}-\boldsymbol{x}_{f_3})\times \boldsymbol{\mathcal{P}}_{r-1}(f_3).
\end{aligned}
$$

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Trimmed local polynomial spaces I

- Let $f \in \Delta_d(\mathcal{M}_h)$, $1 \leq d \leq n$, and integers $\ell \in [0, d]$ and $r \geq 0$ be fixed
- The following direct decompositions hold:

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \mathcal{K}_r^0(f),$$
$$\mathcal{P}_r\Lambda^\ell(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \geq 1$$

- Lowering the polynomial degree of the first component yields trimmed polynomial spaces

$$\mathcal{P}_r^-\Lambda^0(f) \coloneqq \mathcal{P}_r\Lambda^0(f),$$
$$\mathcal{P}_r^-\Lambda^\ell(f) \coloneqq \mathrm{d}\mathcal{P}_r\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \geq 1$$

- the L^2-orthogonal projector onto $\mathcal{P}_r^-\Lambda^k(f)$ is s.t.

$$\forall \omega \in L^2\Lambda^k(f), \quad \int_f \pi_{r,f}^{-,k}\omega \wedge \star\mu = \int_f \omega \wedge \star\mu \quad \forall \mu \in \mathcal{P}_r^-\Lambda^k(f)$$

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Trimmed local polynomial spaces II

Example (Trimmed spaces in dimensions 2 and 3)

Let $n = 3$. For $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$, the vector proxies for trimmed spaces are the Nédélec and Raviart–Thomas spaces:

$$\mathcal{P}_r^- \Lambda^1(f_3) \cong \mathbf{grad}\, \mathcal{P}_r(T) + \mathcal{G}_r^{\rm c}(T) =: \mathcal{N}_r(T),$$
$$\mathcal{P}_r^- \Lambda^2(f_3) \cong \mathbf{curl}\, \mathcal{P}_r(T) + \mathcal{R}_r^{\rm c}(T) =: \mathcal{RT}_r(T).$$

For $F = f_2 \in \Delta_2(\mathcal{M}_h)$, we have

$$\mathcal{P}_r^- \Lambda^1(f_2) \cong \mathbf{rot}\, \mathcal{P}_r(F) + \mathcal{R}_r^{\rm c}(F) =: \mathcal{RT}_r(F).$$

Discrete spaces and interpolators I

- The discrete $H\Lambda^k(\Omega)$ space, $0 \le k \le n$, is

$$\underline{X}^k_{r,h} \coloneqq \bigtimes_{d=k}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \mathcal{P}^-_r\Lambda^{d-k}(f)$$

- Its restrictions to $f \in \Delta_d(\mathcal{M}_h)$, $k \le d \le n$, and ∂f are $\underline{X}^k_{r,f}$ and $\underline{X}^k_{r,\partial f}$
- The components are interpreted as projection of traces on trimmed spaces:

$$\begin{aligned} \underline{I}^k_{r,f} : C^0\Lambda^k(\overline{f}) &\to \underline{X}^k_{r,f} \\ \omega &\mapsto \left(\pi^{-,d'-k}_{r,f'}(\star\, \mathrm{tr}_{f'}\, \omega)\right)_{f'\in\Delta_{d'}(f),\, d'\in[k,d]} \end{aligned}$$

with trace operator $\mathrm{tr}_{f'}$ pullback of the inclusion $f' \hookrightarrow f$

Example (Local polynomial spaces for $n = 3$)

$k \backslash d$	0	1	2	3
0	$\mathbb{R} = \mathcal{P}_r\Lambda^0(f_0)$	$\mathcal{P}_{r-1}\Lambda^1(f_1)$	$\mathcal{P}_{r-1}\Lambda^2(f_2)$	$\mathcal{P}_{r-1}\Lambda^3(f_3)$
1		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{P}_r^-\Lambda^1(f_2)$	$\mathcal{P}_r^-\Lambda^2(f_3)$
2			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{P}_r^-\Lambda^1(f_3)$
3				$\mathcal{P}_r\Lambda^0(f_3)$

$k \backslash d$	0	1	2	3
0	$\mathbb{R} = \mathcal{P}_r(f_0)$	$\mathcal{P}_{r-1}(f_1)$	$\mathcal{P}_{r-1}(f_2)$	$\mathcal{P}_{r-1}(f_3)$
1		$\mathcal{P}_r(f_1)$	$\mathcal{RT}_r(f_2)$	$\mathcal{RT}_r(f_3)$
2			$\mathcal{P}_r(f_2)$	$\mathcal{N}_r(f_3)$
3				$\mathcal{P}_r(f_3)$

Discrete potential and exterior derivative I

- Let $d \in \mathbb{N}$ be s.t. $0 \leq d \leq n$, $f \in \Delta_d(\mathcal{M}_h)$, and notice that

$$\operatorname{tr}_{\partial f} : \Lambda^k(f) \to \Lambda^k(\partial f)$$

- The Stokes formula on f reads: For all $(\omega, \mu) \in C^1\Lambda^k(\bar{f}) \times C^1\Lambda^{d-k-1}(\bar{f})$,

$$\int_f \mathrm{d}\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge \mathrm{d}\mu + \int_{\partial f} \operatorname{tr}_{\partial f} \omega \wedge \operatorname{tr}_{\partial f} \mu$$

- Local reconstructions are obtained emulating this formula

Discrete potential and exterior derivative II

- If $d = k$,

$$P^k_{r,f}\underline{\omega}_f := \star^{-1}\omega_f \in \mathcal{P}_r\Lambda^d(f)$$

- If $k+1 \le d \le n$, we first let, for all $\underline{\omega}_f \in \underline{X}^k_{r,f}$ and all $\mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

then, for all $(\mu, \nu) \in \mathcal{K}^{d-k-1}_{r+1}(f) \times \mathcal{K}^{d-k}_r(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu + (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \nu$$

The case $n = 3$ and $k = 1$ I

- For $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$,

$$\underline{X}^1_{r,f} \cong \underline{\boldsymbol{X}}^r_{\mathbf{curl},T} := \left(\bigtimes_{E \in \mathcal{E}_T} \mathcal{P}_r(E) \right) \times \left(\bigtimes_{F \in \mathcal{F}_T} \boldsymbol{\mathcal{RT}}_r(F) \right) \times \boldsymbol{\mathcal{RT}}_r(T)$$

- Let

$$\underline{\boldsymbol{v}}_T = \left((v_E)_{E \in \mathcal{E}_T}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}, \boldsymbol{v}_T \right) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},T}$$

- The edge tangential trace is simply

$$\gamma^r_{\mathrm{t},E} \underline{\boldsymbol{v}}_E := v_E \quad \forall E \in \mathcal{E}_T$$

The case $n = 3$ and $k = 1$ II

- For all $F \in \mathcal{F}_T$, the face curl is given by: For all $q \in \mathcal{P}_r(F)$,

$$\int_F C_F^r \underline{\boldsymbol{v}}_F \, q = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{\boldsymbol{v}}_E \, q$$

- The face tangential trace is such that, for all $(q, \boldsymbol{w}) \in \mathcal{P}_{r+1}^{\flat}(F) \times \mathcal{R}_r^{\mathrm{c}}(F)$,

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, q + \boldsymbol{w}) = \int_F C_F^r \underline{\boldsymbol{v}}_F \, q + \sum_{E \in \mathcal{E}_F} \varepsilon_{FE} \int_E \gamma_{\mathrm{t},E}^r \underline{\boldsymbol{v}}_E \, q + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w}$$

- The element curl satisfies, for all $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_r(T)$,

$$\int_T \boldsymbol{C}_T^r \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} = \int_T \boldsymbol{v}_T \cdot \mathbf{curl} \, \boldsymbol{w} + \sum_{F \in \mathcal{F}_T} \varepsilon_{TF} \int_F \boldsymbol{\gamma}_{\mathrm{t},F}^r \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w} \times \boldsymbol{n}_F)$$

- Finally, by similar principles, we can construct $\boldsymbol{P}_{\mathbf{curl},T}^r : \underline{\boldsymbol{X}}_{\mathbf{curl},T}^r \to \boldsymbol{\mathcal{P}}_r(T)$

Global discrete exterior derivative and DDR complex

- The spaces $\underline{X}^k_{r,h}$ are connected by the global discrete exterior derivative

$$\underline{\mathrm{d}}^k_{r,h} : \underline{X}^k_{r,h} \to \underline{X}^{k+1}_{r,h}$$
$$\underline{\omega}_h \mapsto \left(\pi^{-,d-k-1}_{r,f}(\star \mathrm{d}^k_{r,f}\underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k+1,n]}$$

- The DDR sequence then reads

$$\underline{X}^0_{r,h} \xrightarrow{\underline{\mathrm{d}}^0_{r,h}} \underline{X}^1_{r,h} \longrightarrow \cdots \longrightarrow \underline{X}^{n-1}_{r,h} \xrightarrow{\underline{\mathrm{d}}^{n-1}_{r,h}} \underline{X}^n_{r,h} \longrightarrow \{0\}$$

- Specifically, for $n = 3$, we recover the complex of [DP and Droniou, 2023]:

$$\underline{X}^r_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^r_h} \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^r_h} \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \xrightarrow{D^r_h} \mathcal{P}_r(\mathcal{T}_h) \longrightarrow \{0\}$$

Complex property

Theorem (Complex property)

For all $0 \le k \le d \le n$ and all $f \in \Delta_d(\mathcal{M}_h)$, it holds,

$$P_{r,f}^k \circ \underline{\mathrm{d}}_{r,f}^{k-1} = \mathrm{d}_{r,f}^{k-1},$$

and, if $d \ge k+1$,

$$\underline{\mathrm{d}}_{r,f}^k \circ \underline{\mathrm{d}}_{r,f}^{k-1} = \underline{0},$$

so that the DDR sequence defines a complex.

Polynomial consistency

Theorem (Polynomial consistency)

For all integers $0 \leq k \leq d \leq n$ and all $f \in \Delta_d(\mathcal{M}_h)$, it holds

$$P^k_{r,f}(\underline{I}^k_{r,f}\omega) = \omega \qquad \forall \omega \in \mathcal{P}_r\Lambda^k(f),$$

and, if $d \geq k+1$,

$$\mathrm{d}^k_{r,f}(\underline{I}^k_{r,f}\omega) = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}^-_{r+1}\Lambda^k(f).$$

Example (The case $(n,d,k) = (3,3,1)$)

The above properties translate as follows for $(n,d,k) = (3,3,1)$:

$$\begin{aligned} \boldsymbol{P}^r_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^r_{\mathbf{curl},T}\boldsymbol{v}) &= \boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_r(T), \\ \boldsymbol{C}^r_T(\underline{\boldsymbol{I}}^r_{\mathbf{curl},T}\boldsymbol{v}) &= \mathbf{curl}\,\boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}_{r+1}(T). \end{aligned}$$

Cohomology I

Theorem (Cohomology of the Discrete de Rham complex)

The cohomology of the DDR complex is isomorphic to that of the continuous de Rham complex.

Example (The case $n = 3$)

For $n = 3$, in terms of vector proxies, this implies, in particular,

$$\text{No "tunnels" crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im} \underline{\boldsymbol{G}}_h^r = \operatorname{Ker} \underline{\boldsymbol{C}}_h^r$$

$$\text{No "voids" contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im} \underline{\boldsymbol{C}}_h^r = \operatorname{Ker} D_h^r$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im} D_h^r = \mathcal{P}_k(\mathcal{T}_h)$$

Cohomology II

Proof.

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \Delta_k^*(\mathcal{M}_h) & \xrightarrow{\partial_k^*} & \Delta_{k+1}^*(\mathcal{M}_h) & \longrightarrow \cdots \\
& \uparrow{\scriptstyle \kappa_k} & & \uparrow{\scriptstyle \kappa_{k+1}} & \\
\cdots \longrightarrow & \underline{X}_{0,h}^k & \xrightarrow{\mathrm{d}_{0,h}^k} & \underline{X}_{0,h}^{k+1} & \longrightarrow \cdots \\
& {\scriptstyle \underline{R}_h^k} \uparrow \ \downarrow {\scriptstyle \underline{E}_h^k} & & {\scriptstyle \underline{R}_h^{k+1}} \uparrow \ \downarrow {\scriptstyle \underline{E}_h^{k+1}} & \\
\cdots \longrightarrow & \underline{X}_{r,h}^k & \xrightarrow{\mathrm{d}_{r,h}^k} & \underline{X}_{r,h}^{k+1} & \longrightarrow \cdots
\end{array}
$$

Key point: design of the extension cochain map $\underline{E}_h$ □

Poincaré inequalities

Theorem (Poincaré inequalities)

For any k, denote by $|||\cdot|||_{k,h}$ the L^2-like discrete component norm on $\underline{X}^k_{r,h}$.
Let, additionally, $\langle\cdot,\cdot\rangle_{k,h}$ be a scalar product on $\underline{X}^k_{r,h}$ inducing a norm $\simeq |||\cdot|||_{k,h}$.
Then, denoting by $(\bullet)^\perp$ the orthogonal complement of $\bullet$ for $\langle\cdot,\cdot\rangle_{k,h}$, it holds:

$$|||\underline{\omega}_h|||_{k,h} \lesssim |||\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h|||_{k+1,h} \qquad \forall \underline{\omega}_h \in (\mathrm{Ker}\, \underline{\mathrm{d}}^k_{r,h})^\perp.$$

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Discrete L^2-product

- For all $0 \leq k \leq n$, we let $(\cdot,\cdot)_{k,h} : \underline{X}^k_{r,h} \times \underline{X}^k_{r,h} \to \mathbb{R}$ be s.t.

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} (\underline{\omega}_f, \underline{\mu}_f)_{k,f}$$

  with

$$(\underline{\omega}_f, \underline{\mu}_f)_{k,f} \coloneqq \int_f P^k_{r,f}\underline{\omega}_f \wedge \star P^k_{r,f}\underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \quad \forall f \in \Delta_n(\mathcal{M}_h)$$

- Above, $s_{k,f}$ is a stabilization contribution s.t., with h_f diameter of f,

$$s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) = \sum_{d'=k}^{n-1} h_f^{n-d'} \sum_{f' \in \Delta_{d'}(f)} \int_{f'} (\mathrm{tr}_{f'} P^k_{r,f}\underline{\omega}_f - P^k_{r,f'}\underline{\omega}_{f'}) \wedge \star(\mathrm{tr}_{f'} P^k_{r,f}\underline{\mu}_f - P^k_{r,f'}\underline{\mu}_{f'})$$

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Outline

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Discrete problem

- With, for simplicity, $b_1 = b_2 = 0$ and $\mu \in \mathbb{R}$, consider the problem: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{A}\, \mathrm{div}\, \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- Its DDR discretization reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$ s.t.

$$\begin{aligned} (\mu \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h}, \\ (\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{\boldsymbol{A}}_h\, D^r_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \end{aligned}$$

- For $b_2 \neq 0$, we need to add orthogonality to harmonic forms

Stability

Theorem (Stability)

Define the bilinear form $\mathcal{A}_h : \left[\underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}\right]^2 \to \mathbb{R}$ *s.t.*

$$\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) := (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{\boldsymbol{u}}_h \, D^r_h \underline{\boldsymbol{v}}_h.$$

Then, the following inf-sup condition holds: $\forall (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h}$,

$$|||(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)|||_h \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}^r_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^r_{\mathrm{div},h} \setminus \{(\underline{\mathbf{0}}, \underline{\mathbf{0}})\}} \frac{\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h}$$

with $|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h^2 := \|\underline{\boldsymbol{\tau}}_h\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^r_h \underline{\boldsymbol{\tau}}_h\|^2_{\mathrm{div},h} + \|\underline{\boldsymbol{v}}_h\|^2_{\mathrm{div},h} + \|D^r_h \underline{\boldsymbol{v}}_h\|^2_{L^2(\Omega)}$.

Proof.

Analogous to the continuous case! □

Error estimate

Theorem (Error estimate for the magnetostatics problem)

Assume $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{r+2}(\mathcal{T}_h)^3$ and $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \cap H^{r+2}(\mathcal{T}_h)^3$. Then, we have the following error estimate:

$$|||(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^r_{\mathbf{curl},h}\boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^r_{\mathrm{div},h}\boldsymbol{A})|||_h \lesssim h^{r+1}.$$

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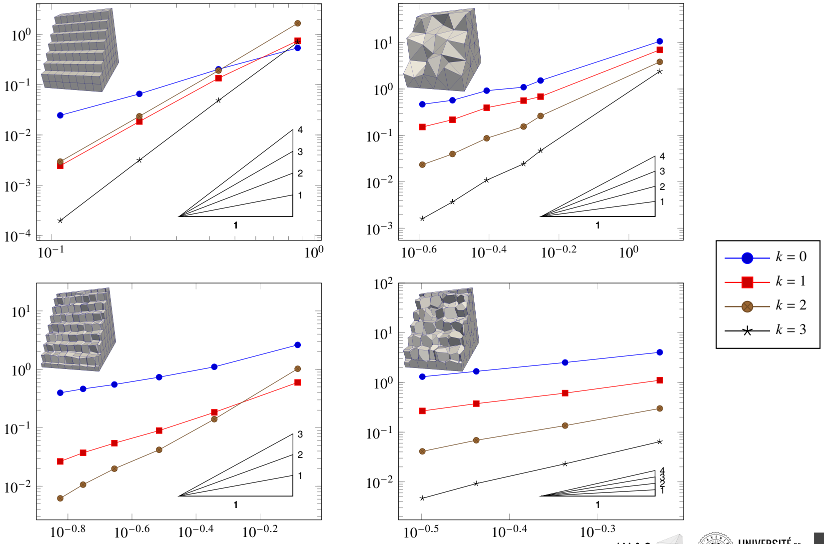
Convergence: Energy error vs. meshsize



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Thank you for your attention!

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