Basic principles of polytopal approximations of partial differential equations

Daniele A. Di Pietro

Scuola Superiore Meridionale
Napoli, 02/05/2024

Outline

Outline

Setting

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal domain
- We focus on the Poisson problem: Given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Let $f \in L^2(\Omega)$. A possible weak formulation reads: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega)$$

- The well-posedness of this problem hinges on the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)^d} \quad \forall v \in H_0^1(\Omega)$$

Local polynomial spaces and L^2-orthogonal projector

- Denote by $\mathbb{P}_d^\ell$ the space of d-variate polynomials of total degree $\leq \ell$
- Let $Y \subset \mathbb{R}^d$ and $\mathcal{P}^\ell(Y) := \left\{\text{restriction of } \mathbb{P}_d^\ell \text{ to } Y\right\}$
- The L^2-orthogonal projector $\pi_{\mathcal{X}} v$ on a subspace $\mathcal{X} \subset L^2(Y)$ is s.t.

$$\pi_{\mathcal{X}} v = \arg\min_{w \in \mathcal{X}} \|v - w\|_{L^2(Y)}^2 \iff \int_Y (v - \pi_{\mathcal{X}} v) w = 0 \quad \forall w \in \mathcal{X}$$

- Under mild assumptions on Y and for all Sobolev seminorms:
  - $\pi_{\mathcal{P}^\ell(Y)} v$ approximates v
  - if $\ell \geq 1$, $\pi_{\nabla \mathcal{P}^\ell(Y)} \nabla v$ approximates ∇v

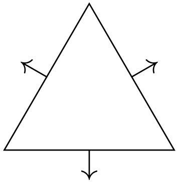
Local polynomial spaces and L^2-orthogonal projector

- Denote by $\mathbb{P}_d^\ell$ the space of d-variate polynomials of total degree $\leq \ell$
- Let $Y \subset \mathbb{R}^d$ and $\mathcal{P}^\ell(Y) := \left\{\text{restriction of } \mathbb{P}_d^\ell \text{ to } Y\right\}$
- The L^2-orthogonal projector $\pi_{\mathcal{X}} v$ on a subspace $\mathcal{X} \subset L^2(Y)$ is s.t.

$$\pi_{\mathcal{X}} v = \arg\min_{w \in \mathcal{X}} \|v - w\|_{L^2(Y)}^2 \iff \int_Y (v - \pi_{\mathcal{X}} v) w = 0 \quad \forall w \in \mathcal{X}$$

- Under mild assumptions on Y and for all Sobolev seminorms:
  - $\pi_{\mathcal{P}^\ell(Y)} v$ approximates v
  - if $\ell \geq 1$, $\pi_{\nabla \mathcal{P}^\ell(Y)} \nabla v$ approximates ∇v
- The details can be found in [DP and Droniou, 2020, Chapter 1]

The Raviart–Thomas–Nédélec element

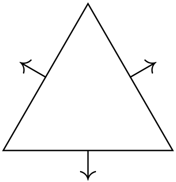


- Denote by T a d-simplex and by $\mathcal{F}_T$ the set collecting its faces
- Let $\mathcal{RTN}^1(T) \coloneqq \mathcal{P}^0(T)^d + x\mathcal{P}^0(T)$
- Define the degrees of freedom $\sigma \coloneqq (\sigma_F)_{F\in\mathcal{F}_T}$ s.t.

$$\sigma_F : \mathcal{RTN}^1(T) \ni \tau \mapsto \frac{1}{|F|}\int_F \tau\cdot n_{TF} \in \mathbb{R}$$

- $(T, \mathcal{RTN}^1(T), \sigma)$ is a FE [Raviart and Thomas, 1977, Nédélec, 1980]

The Raviart–Thomas–Nédélec element

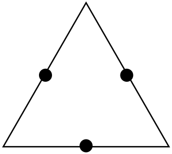


- Denote by T a d-simplex and by $\mathcal{F}_T$ the set collecting its faces
- Let $\mathcal{RTN}^1(T) := \mathcal{P}^0(T)^d + x\mathcal{P}^0(T)$
- Define the degrees of freedom $\sigma := (\sigma_F)_{F\in\mathcal{F}_T}$ s.t.

$$\sigma_F : \mathcal{RTN}^1(T) \ni \tau \mapsto \frac{1}{|F|}\int_F \tau\cdot n_{TF} \in \mathbb{R}$$

- $(T, \mathcal{RTN}^1(T), \sigma)$ is a FE [Raviart and Thomas, 1977, Nédélec, 1980]
- Notice that $\tau_{|F}\cdot n_{TF} \in \mathcal{P}^0(F)$ for all $\tau \in \mathcal{RTN}^1(T)$ and all $F \in \mathcal{F}_T$

The Crouzeix–Raviart element

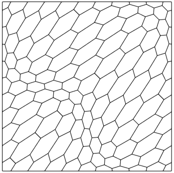


- Let now $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom $\sigma := (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : \mathcal{P}^1(T) \ni v \mapsto v_F := \frac{1}{|F|} \int_F v \in \mathbb{R}$$

- $(T, \mathcal{P}^1(T), \sigma)$ is a FE [Crouzeix and Raviart, 1973]

A magic formula



- Let $(\mathcal{T}_h, \mathcal{F}_h)$ be a mesh of Ω with $\mathcal{T}_h$ collecting elements and $\mathcal{F}_h$ faces
- Denote by $(v_F)_{F \in \mathcal{F}_h}$ a family of functions with

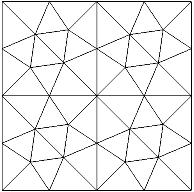
$$v_F \in L^2(F) \text{ for all } F \in \mathcal{F}_h \text{ and } v_F = 0 \text{ if } F \subset \partial\Omega$$

- Then, for all $\tau \in H(\mathrm{div}; \Omega)$ slightly smoother inside each element,

$$\boxed{\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) = 0} \qquad \text{(magic)}$$

Outline

A non-conforming finite element scheme I



- Let $\mathcal{T}_h$ be a conforming simplicial mesh of Ω
- Let $\mathcal{CR}(\mathcal{T}_h)$ be the Crouzeix–Raviart space on $\mathcal{T}_h$ and set

$$\mathcal{CR}_0(\mathcal{T}_h) := \{v_h \in \mathcal{CR}(\mathcal{T}_h) \,:\, v_F = 0 \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \,\} \not\subset H_0^1(\Omega)$$

- With ∇_h broken gradient, the scheme reads: Find $u_h \in \mathcal{CR}_0(\mathcal{T}_h)$ s.t.

$$a_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in \mathcal{CR}_0(\mathcal{T}_h)$$

Stability analysis I

Lemma (Discrete Poincaré inequality in the Crouzeix–Raviart space)

For all $v_h \in \mathcal{CR}_0(\mathcal{T}_h)$,

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

where $a \lesssim b$ *means* $a \le Cb$ *with* C *independent of* h.

- For all $T \in \mathcal{T}_h$, the Poincaré–Wirtinger inequality gives, with $v_T := (v_h)_{|T}$,

$$\|v_T\|_{L^2(T)} \lesssim \|\pi_{\mathcal{P}^0(T)} v_T\|_{L^2(T)} + h_T \|\nabla v_T\|_{L^2(T)^d}$$

- Hence, letting $\overline{v}_h := \pi_{\mathcal{P}^0(\mathcal{T}_h)} v_h$, it suffices to prove that

$$\|\overline{v}_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

- Since $\mathrm{div} : \mathcal{RTN}^1(\mathcal{T}_h) \to \mathcal{P}^0(\mathcal{T}_h)$ is surjective, $\exists \tau_h \in \mathcal{RTN}^1(\mathcal{T}_h)$ s.t.

$$\mathrm{div}\, \tau_h = \overline{v}_h \text{ and } \|\tau_h\|_{H(\mathrm{div};\Omega)} \lesssim \|\overline{v}_h\|_{L^2(\Omega)}$$

Stability analysis II

- We can then write

$$\|\overline{v}_h\|^2_{L^2(\Omega)} = \sum_{T\in\mathcal{T}_h} \int_T \cancel{\pi_{\mathcal{P}^0(T)}} v_T \overbrace{\operatorname{div}\tau_h}^{\in\mathcal{P}^0(T)}$$

$$\overset{\text{IBP}}{=} -\int_\Omega \nabla_h v_h \cdot \tau_h + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F {\color{red}v_T}\,\underbrace{(\tau_h\cdot n_{TF})}_{\in\mathcal{P}^0(F)}$$

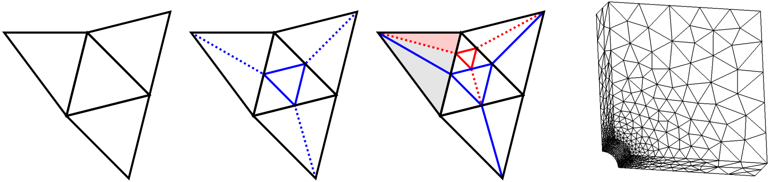
- Replacing ${\color{red}v_T \leftarrow \pi_{\mathcal{P}^0(F)} v_T =: v_F}$ in the boundary term and rearranging,

$$\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F {\color{red}v_T}(\tau_h\cdot n_{TF}) = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F {\color{red}v_F}(\tau_h\cdot n_{TF}) \overset{\text{(magic)}}{=} 0$$

- Using Cauchy–Schwarz inequalities, we get

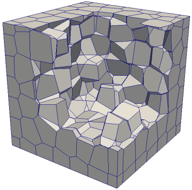
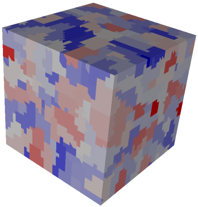
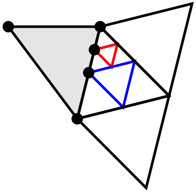
$$\|\overline{v}_h\|^2_{L^2(\Omega)} \le \|\nabla_h v_h\|_{L^2(\Omega)^d}\|\tau_h\|_{L^2(\Omega)^d} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}\|\overline{v}_h\|_{L^2(\Omega)}$$

Limitations of the finite element approach



- Approach limited to conforming meshes with standard elements
  - ⟹ Local refinement requires to trade mesh size for mesh quality
  - ⟹ Complex geometries may require a large number of elements
  - ⟹ The element shape cannot be adapted to the solution
- Treating more general meshes in the FE spirit would significantly increase the space dimension [Droniou et al., 2021]
- The extension to high-order is also not straightforward

Fully discrete polytopal approach

- Key idea: replace both spaces and operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Several strategies to reduce the number of unknowns on general shapes
- Elegant analysis framework available

A few key references

- Introduction of Hybrid High-Order (HHO) methods [DP et al., 2014]
- Fully discrete analysis framework [DP and Droniou, 2018]
- A monograph on HHO methods [DP and Droniou, 2020]
- Introduction of Discrete de Rham (DDR) methods [DP et al., 2020]
- DDR for the de Rham complex of differential forms [Bonaldi, DP et al, 2023]

Outline

A paradigm shift

- Let $v_T \in \mathcal{P}^1(T)$ and set $v_F := \pi_{\mathcal{P}^0(F)}(v_T)_{|F}$ for all $F \in \mathcal{F}_T$ as before
- We have $\nabla v_T \in \mathcal{P}^0(T)^d$ and, for all $\tau \in \mathcal{P}^0(T)^d$,

$$\int_T \nabla v_T \cdot \tau = -\int_T v_T \cancel{\operatorname{div} \tau} + \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF})$$

- Moreover, with $\overline{x}_Y$ center of mass of $Y \in \{T\} \cup \mathcal{F}_T$, noticing that

$$\pi_{\mathcal{P}^0(T)} v = v_T(\overline{x}_T) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} v_T(\overline{x}_F) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F,$$

where we have used linearity and $\overline{x}_T = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \overline{x}_F$

A paradigm shift

- Let $v_T \in \mathcal{P}^1(T)$ and set $v_F \coloneqq \pi_{\mathcal{P}^0(F)}(v_T)_{|F}$ for all $F \in \mathcal{F}_T$ as before
- We have $\nabla v_T \in \mathcal{P}^0(T)^d$ and, for all $\tau \in \mathcal{P}^0(T)^d$,

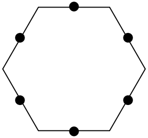
$$\int_T \nabla v_T \cdot \tau = -\int_T v_T \cancel{\operatorname{div} \tau} + \sum_{F \in \mathcal{F}_T} \int_F v_T(\tau \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F v_F(\tau \cdot n_{TF})$$

- Moreover, with $\overline{x}_Y$ center of mass of $Y \in \{T\} \cup \mathcal{F}_T$, noticing that

$$\pi_{\mathcal{P}^0(T)} v = v_T(\overline{x}_T) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} v_T(\overline{x}_F) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F,$$

  where we have used linearity and $\overline{x}_T = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \overline{x}_F$
- **These formulas remain valid when T is a general polytope!**

Generalization to polytopes



- Define the following space spanned by vectors of polynomials:

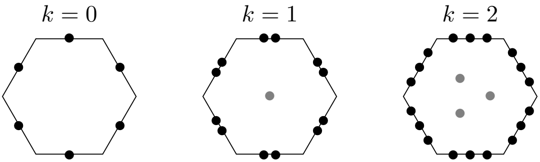
$$\underline{V}_T^0 := \left\{ \underline{v}_T := (v_F)_{F \in \mathcal{F}_T} \ : \ v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

- We can define a potential reconstruction $r_T^1 : \underline{V}_T^0 \to \mathcal{P}^1(T)$ enforcing

$$\int_T \nabla r_T^1 \underline{v}_T \cdot \tau = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^0(T)^d$$

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = v_T := \frac{1}{\text{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

Extension to arbitrary-order



- Let $k \geq 0$ and define the Hybrid High-Order (HHO) space

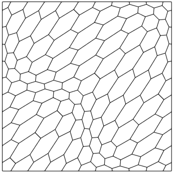
$$\underline{V}_T^k := \Big\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \ : \\ v_T \in \mathcal{P}^{k-1}(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \Big\}$$

- We define $r_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^k(T)^d$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T \nabla r_T^{k+1} \underline{v}_T \cdot \nabla w = -\int_T v_T \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \pi_{\mathcal{P}^0(T)} v_T$$

Global HHO space and H^1-like seminorm



- Given a polytopal mesh $\mathcal{T}_h$ of Ω, define the global HHO space

$$\underline{V}_h^k := \Big\{\underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_T}) \ : \\ v_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h\Big\}$$

- We define on $\underline{V}_h^k$ the H^1-like seminorm

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

where $\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2$ for all $T \in \mathcal{T}_h$

Discrete Poincaré inequality in HHO spaces I

Lemma (Discrete Poincaré inequality in HHO spaces)

Denote by $\underline{V}_{h,0}^k$ the subspace of $\underline{V}_h^k$ with vanishing boundary values. For any $\underline{v}_h \in \underline{V}_{h,0}^k$, letting $v_h \in \mathcal{P}^{\max(k-1,0)}(\mathcal{T}_h)$ be s.t. $(v_h)_{|T} := v_T$ for all $T \in \mathcal{T}_h$,

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h},$$

hence $\|\cdot\|_{1,h}$ is a norm on $\underline{V}_{h,0}^k$.

- We can no longer lift $\overline{v}_h = \pi_{\mathcal{P}^0(\mathcal{T}_h)} v_h \in \mathcal{P}^0(\mathcal{T}_h)$ in $\operatorname{div} \mathcal{RTN}^1(\mathcal{T}_h)$!
- But $\operatorname{div} : H^1(\Omega)^d \to L^2(\Omega)$ is surjective, so there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = \overline{v}_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|\overline{v}_h\|_{L^2(\Omega)}$$

Discrete Poincaré inequality in HHO spaces II

- Let us take over from

$$\|\overline{v}_h\|_{L^2(\Omega)}^2 \overset{\text{(magic)}}{=} -\int_T \nabla_h v_h \cdot \tau + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F (v_T - v_F)(\tau\cdot n_{TF})$$

$$\overset{\text{C–S}}{\leq} \|\underline{v}_h\|_{1,h}\left[\sum_{T\in\mathcal{T}_h}\left(\|\tau\|_{L^2(T)^2}^2 + h_T\|\tau\|_{L^2(\partial T)}^2\right)\right]^{\frac{1}{2}}$$

- Using Cauchy–Schwarz and trace inequalities along with $h_T \leq h_\Omega \lesssim 1$,

$$\|\overline{v}_h\|_{L^2(\Omega)}^2 \lesssim \|\underline{v}_h\|_{1,h}\|\tau\|_{H^1(T)^d} \lesssim \|\underline{v}_h\|_{1,h}\|\overline{v}_h\|_{L^2(\Omega)}$$

- **The novelty is that $\|\underline{v}_h\|_{1,h}$ replaces $\|\nabla_h v_h\|_{L^2(\Omega)^d}$!**

An HHO scheme

We consider the following scheme: Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

where, for all $T \in \mathcal{T}_h$,

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla r_T^{k+1} \underline{u}_T \cdot \nabla r_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

and the symmetric semi-definite stabilization bilinear form s_T satisfies

$$\boxed{\|\underline{v}_T\|_{1,T} \lesssim a_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \lesssim \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{V}_T^k} \tag{ST1}$$

Stability analysis

Lemma (Well-posedness of the HHO discrete problem)

The HHO problem admits a unique solution that satisfies

$$\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}.$$

- Squaring and summing (ST1) over $T \in \mathcal{T}_h$, we have

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k =: a_h \text{ is uniformly coercive}$$

- Using the Cauchy–Schwarz and discrete Poincaré inequalities,

$$\int_\Omega f v_h \le \|f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \|\underline{v}_h\|_{1,h}$$

- Letting $\underline{v}_h = \underline{u}_h$ above, the a priori estimate follows

Error analysis I

- Let $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ be the HHO interpolator s.t.

$$\underline{I}_h^k v := ((\pi_{\mathcal{P}^{k-1}(T)} v)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P}^k(F)} v)_{F \in \mathcal{F}_h}) \quad \forall v \in H^1(\Omega)$$

- We aim at estimating the error

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u \in \underline{V}_{h,0}^k$$

- The error satisfies the following equation: For all $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$a_h(\underline{e}_h, \underline{v}_h) = a_h(\underline{u}_h, \underline{v}_h) - a_h(\underline{I}_h^k u, \underline{v}_h) = \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(u; \underline{v}_h)$$

- A straightforward modification of the stability proof gives

$$\|\underline{e}_h\|_{1,h} \le \sup_{\underline{v}_h \in \underline{V}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

Error analysis II

We reformulate the components of the consistency error $\mathcal{E}_h(\underline{v}_h)$:

$$\int_\Omega f v_h = -\sum_{T\in\mathcal{T}_h}\int_T \Delta u\, v_h$$

$$\overset{\text{IBP, (magic)}}{=} \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla u\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla u\cdot n_{TF})(v_F - v_T)\right]$$

$$a_h(\underline{I}_h^k u, \underline{v}_h) = \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla r_T^{k+1}(\underline{I}_T^k u)\cdot\nabla r_T^{k+1}\underline{v}_T + s_T(\underline{I}_T^k u, \underline{v}_T)\right]$$

$$= \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla r_T^{k+1}(\underline{I}_T^k u)\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla r_T^{k+1}(\underline{I}_T^k u)\cdot n_{TF})(v_F - v_T)\right]$$

$$+ \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)$$

Error analysis III

Gathering the above results, we get

$$\mathcal{E}_h(\underline{v}_h) = \underbrace{\sum_{T\in\mathcal{T}_h}\int_T \left[\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)\right]\cdot\nabla v_T}_{\mathfrak{T}_1}$$
$$+ \underbrace{\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \left[\nabla u - (\nabla r_T^{k+1}(\underline{I}_T^k u)\cdot n_{TF})\right](v_F - v_T)}_{\mathfrak{T}_2}$$
$$\underbrace{-\sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)}_{\mathfrak{T}_3}$$

Approximation properties of the potential reconstruction I

- By definition, for all $T \in \mathcal{T}_h$, all $v \in H^1(T)$, and all $w \in \mathcal{P}^{k+1}(T)$,

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = -\int_T \pi_{\mathcal{P}^{k-1}(T)} v \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_{\mathcal{P}^k(F)} v (\nabla w \cdot n_{TF})$$

- Noticing that $\Delta w \in \mathcal{P}^{k-1}(T)$ and $\nabla w \cdot n_{TF} \in \mathcal{P}^k(F)$, we can remove the projectors and integrate by parts to obtain

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = \int_T \nabla v \cdot \nabla w \quad \forall w \in \mathcal{P}^{k+1}(T)$$

- This shows that $\nabla r_T^{k+1} \circ \underline{I}_T^k = \pi_{\nabla \mathcal{P}^{k+1}(T)} \circ \nabla$

Approximation properties of the potential reconstruction II

- Noticing that $\nabla v_T \in \nabla\mathcal{P}^k(T) \subset \nabla\mathcal{P}^{k+1}(T)$, we can write, for all $T \in \mathcal{T}_h$,

$$\int_T \left[\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)\right] \cdot \nabla v_T = \int_T \left[\nabla u - \pi_{\nabla\mathcal{P}^{k+1}(T)}(\nabla u)\right] \cdot \nabla v_T = 0,$$

  hence

$$\boxed{\mathfrak{T}_1 = 0}$$

- Using Cauchy–Schwarz inequalities and the definition of $\|\cdot\|_{1,h}$,

$$\mathfrak{T}_2 \le \left(\sum_{T\in\mathcal{T}_h} h_T \|\nabla u - \pi_{\nabla\mathcal{P}^{k+1}(T)}(\nabla u)\|_{\partial T}^2\right)^{\frac{1}{2}} \|\underline{v}_h\|_{1,h}$$

- If, additionally, $u \in H^{k+2}(\mathcal{T}_h)$,

$$\boxed{\mathfrak{T}_2 \lesssim h^{k+1}|u|_{H^{k+2}(\mathcal{T}_h)}\|\underline{v}_h\|_{1,h}}$$

Polynomial consistency of the stabilization I

- To have $\mathfrak{T}_3$ scale as $\mathfrak{T}_2$, we further assume polynomial consistency:

$$\boxed{s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k} \tag{ST2}$$

- For all $v \in H^{k+2}(T)$, setting $|\cdot|_{s,T} := s_T(\cdot,\cdot)^{\frac{1}{2}}$, we have

$$\begin{aligned}
|\underline{I}_T^k v|_{s,T} &\overset{\text{(ST2)}}{=} \min_{w \in \mathcal{P}^{k+1}(T)} |\underline{I}_T^k (v - w)|_{s,T} \\
&\overset{\text{(ST1)}}{\lesssim} \min_{w \in \mathcal{P}^{k+1}(T)} \|\underline{I}_T^k (v - w)\|_{1,T} \lesssim h_T^{k+1} |v|_{H^{k+2}(T)}
\end{aligned}$$

hence, by Cauchy–Schwarz inequalities and again (ST1),

$$\boxed{\mathfrak{T}_3 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}}$$

Polynomial consistency of the stabilization II

Theorem (Error estimate for the HHO scheme)

Denote by $u \in H^1_0(\Omega)$ the solution to the Poisson problem and by $\underline{u}_h \in \underline{V}_h^k$ its HHO approximation. Then, under (ST1)–(ST2)*, and further assuming $u \in H^{k+2}(\mathcal{T}_h)$, it holds*

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)}.$$

An example of stabilization bilinear form

Example

Let, for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{V}_T^k$,

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) := \underline{v}_T - \underline{I}_T^k (r_T^{k+1} \underline{v}_T).$$

The stabilization bilinear form

$$s_T(\underline{w}_T, \underline{v}_T) := h_T^{-2} \int_T \delta_T^k \underline{w}_T \, \delta_T^k \underline{v}_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \delta_{TF}^k \underline{w}_T \, \delta_{TF}^k \underline{v}_T$$

satisfies properties (ST1)–(ST2).

Numerical example

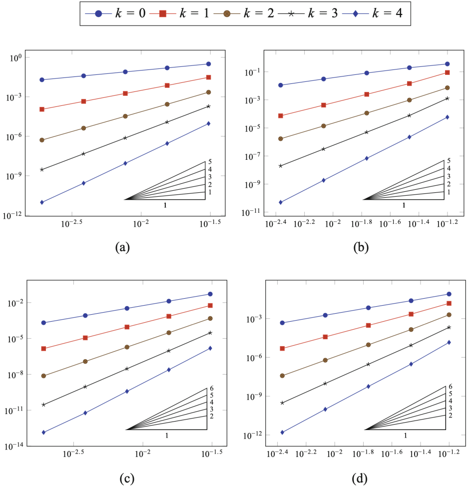
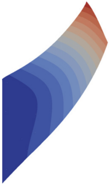
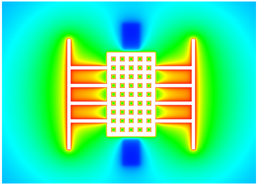


Figure: $\|\underline{e}_h\|_{1,h}$ (top) and $\|e_h\|_{L^2(\Omega)}$ (bottom) as functions of h for uniformly refined triangular (left) and hexagonal (right) mesh families

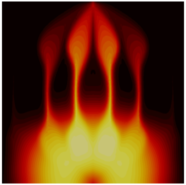
Examples of applications



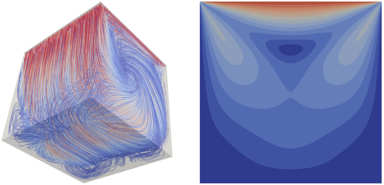
Solid mechanics



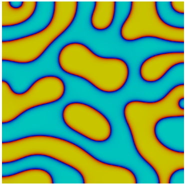
Electromagnetism



Porous media



Fluid mechanics



Phase separation

Funded by the European Union

European Research Council
Established by the European Commission

NEMESIS
New generation methods
for numerical simulations

Funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Thank you for your attention!

`https://erc-nemesis.eu/events/workshop-montpellier`

References I

Bonaldi, F., Di Pietro, D. A., Droniou, J., and Hu, K. (2023).
An exterior calculus framework for polytopal methods.
http://arxiv.org/abs/2303.11093.

Crouzeix, M. and Raviart, P.-A. (1973).
Conforming and nonconforming finite element methods for solving the stationary Stokes equations.
RAIRO Modél. Math. Anal. Num., 7(3):33–75.

Di Pietro, D. A. and Droniou, J. (2018).
A third Strang lemma for schemes in fully discrete formulation.
Calcolo, 55(40).

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order method for polytopal meshes.
Number 19 in Modeling, Simulation and Application. Springer International Publishing.

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Meth. Appl. Math., 14(4):461–472.

Droniou, J., Eymard, R., Gallouët, T., and Herbin, R. (2021).
Polyhedral Methods in Geosciences, chapter Non-conforming finite elements on polytopal meshes, pages 1–35.
Number 27 in SEMA-SIMAI. Springer.

Nédélec, J.-C. (1980).
Mixed finite elements in $\mathbf{R}^3$.
Numer. Math., 35(3):315–341.

References II

Raviart, P. A. and Thomas, J. M. (1977).
A mixed finite element method for 2nd order elliptic problems.
In Galligani, I. and Magenes, E., editors, *Mathematical Aspects of the Finite Element Method*. Springer, New York.