Polytopal approximations of Hilbert complexes

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NEMESIS

New generation methods for numerical simulations

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Research Cluster 1: Discrete Hilbert complexes

- Polytopal de Rham complexes
  - Construction of a Rosetta stone to bridge the virtual and fully discrete approaches
  - Development of novel polytopal de Rham complexes with competitive features
  - Application of these new complexes to model problems
  - Development of **Polytopal Exterior Calculus (PEC)**
- Extended polytopal complexes
  - Novel polytopal elasticity complexes, applied to solid- and fluid-mechanics
  - Novel polytopal Hessian complexes, applied to linear elasticity and general relativity
  - Generalization of PC to cover extended polytopal complexes

Outline

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Setting

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- $b_0 = 1$ (number of connected components) and $b_3 = 0$ (since $d = 3$)
- b_1 and b_2 respectively account for the number of tunnels and voids

$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$

$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$

- Assume for the moment that Ω has trivial topology, i.e.,

$$b_1 = b_2 = 0$$

Two model problems

- We consider two examples of PDEs set in $H^1(\Omega)$, $H(\text{curl};\Omega)$, and $H(\text{div};\Omega)$
- The Stokes problem: Find $(u,p) \in H(\text{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} \nu \int_\Omega \text{curl}\, u \cdot \text{curl}\, v + \int_\Omega \text{grad}\, p \cdot v &= \int_\Omega f \cdot v \quad && \forall v \in H(\text{curl};\Omega), \\ -\int_\Omega u \cdot \text{grad}\, q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

- The magnetostatics problem: Find $(H,A) \in H(\text{curl};\Omega) \times H(\text{div};\Omega)$ s.t.

$$\begin{aligned} \mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \text{curl}\, \tau &= 0 && \forall \tau \in H(\text{curl};\Omega), \\ \int_\Omega \text{curl}\, H \cdot v + \int_\Omega \text{div}\, A\, \text{div}\, v &= \int_\Omega J \cdot v && \forall v \in H(\text{div};\Omega) \end{aligned}$$

A unified view

- The above problems are mixed formulations involving two fields:
  Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma,v) + c(u,v) &= g(v) \quad \forall v \in U, \end{aligned}$$

  or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma,u),(\tau,v)) = f(\tau) + g(v) \qquad \forall (\tau,v) \in \Sigma \times U$$

  with

$$\mathcal{A}((\sigma,u),(\tau,v)) := a(\sigma,\tau) + b(\tau,u) - b(\sigma,v) + c(u,v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties, possibly depending on the topology of Ω:

$$\text{Im}\,\text{grad} \subset \text{Ker}\,\text{curl},$$
$$\text{Im}\,\text{curl} \subset \text{Ker}\,\text{div},$$
$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \text{Im}\,\text{div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

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- Key properties, possibly depending on the topology of Ω:

no tunnels crossing Ω ($b_1 = 0$) $\implies$ $\text{Im}\,\text{grad} = \text{Ker}\,\text{curl}$ (Stokes)

no voids contained in Ω ($b_2 = 0$) $\implies$ $\text{Im}\,\text{curl} = \text{Ker}\,\text{div}$ (magnetostatics)

$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\implies$ $\text{Im}\,\text{div} = L^2(\Omega)$ (magnetostatics)

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- Key properties, possibly depending on the topology of Ω:

$$\begin{aligned} \text{no tunnels crossing } \Omega\ (b_1 = 0) &\implies \text{Im grad} = \text{Ker curl} \quad \text{(Stokes)} \\ \text{no voids contained in } \Omega\ (b_2 = 0) &\implies \text{Im curl} = \text{Ker div} \quad \text{(magnetostatics)} \\ \Omega \subset \mathbb{R}^3\ (b_3 = 0) &\implies \text{Im div} = L^2(\Omega) \quad \text{(magnetostatics)} \end{aligned}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

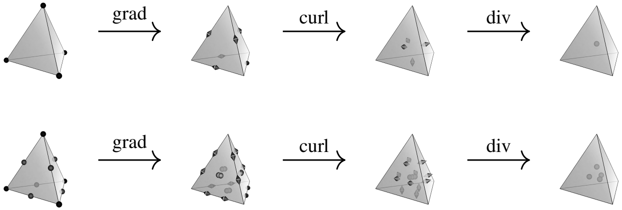
- **Emulating these algebraic properties is key for stable discretizations**

The Finite Element way

- Trimmed FE complexes on a tetrahedron T[1]: For any $k \geq 1$

$k = 1$

$k = 2$



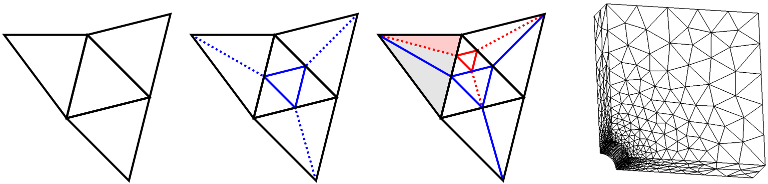
$\dots$

- On a conforming tetrahedral meshes $\mathcal{T}_h$, these spaces can be glued together

$$
\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{P}^k_{\text{c}}(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

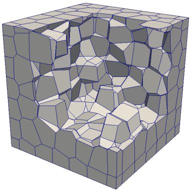
[1] [Raviart and Thomas, 1977, Nédélec, 1980]

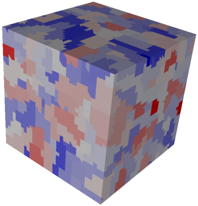
Limitations

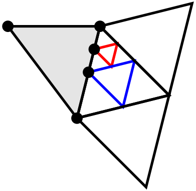


- Approach limited to conforming meshes with standard elements
  - ⟹ Local refinement requires to trade mesh size for mesh quality
  - ⟹ Complex geometries may require a large number of elements
  - ⟹ The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward

Polytopal approaches







- Key idea: replace spaces and, possibly, operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Higher-level point of view, possibly resulting in leaner constructions
- Several strategies to reduce the number of unknowns on general shapes
- Agglomeration-based[2] techniques (see P. Antonietti's presentation)

[2] [Bassi et al., 2012], [Antonietti et al., 2013]

Outline

Isomorphism in cohomology

Theorem (Complexes with isomorphic cohomologies[3])

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & V_i & \xrightarrow{d_i} & V_{i+1} & \longrightarrow & \cdots \\
 & & E_i \downarrow \uparrow R_i & & E_{i+1} \downarrow \uparrow R_{i+1} & & \\
\cdots & \longrightarrow & W_i & \xrightarrow{\partial_i} & W_{i+1} & \longrightarrow & \cdots
\end{array}
$$

Assume that reduction R and extension E are s.t., for all i,

- $R_i E_i = \mathrm{Id}_{W_i}$;
- $(E_{i+1} R_{i+1} - \mathrm{Id}_{V_{i+1}}) \operatorname{Ker} d_{i+1} \subset \operatorname{Im} d_i$;
- $\partial_i E_i = E_{i+1} d_i$ and $d_i R_i = R_{i+1} \partial_i$.

Then, the sequences are complexes with isomorphic cohomologies.

[3] [DP, Droniou and Pitassi, 2023]

Cohomology of the trimmed FE complex

- Let $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ be the simplicial complex underlying a FE mesh
- Denoting by ∂ the coboundary operator, we have

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h}\,\cong\uparrow & & \kappa_{1,h}\,\cong\uparrow & & \kappa_{2,h}\,\cong\uparrow & & \kappa_{3,h}\,\cong\uparrow \\
\mathcal{P}_{\rm c}^1(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^0(\mathcal{T}_h) \\
\downarrow\uparrow I_{\mathrm{grad},h}^1 & & \downarrow\uparrow I_{\mathrm{curl},h}^1 & & \downarrow\uparrow I_{\mathrm{div},h}^1 & & \downarrow\uparrow \pi_h^0 \\
\mathcal{P}_{\rm c}^k(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

  with κ_h de Rham map, $I_{\bullet,h}^1$ interpolator, and π_h^0 L^2-orthogonal projector
- By the de Rham Theorem, the first two rows have isomorphic cohomologies

Cohomology of the trimmed FE complex

- Let $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ be the simplicial complex underlying a FE mesh
- Denoting by ∂ the coboundary operator, we have

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h}\,\cong\,\big\uparrow & & \kappa_{1,h}\,\cong\,\big\uparrow & & \kappa_{2,h}\,\cong\,\big\uparrow & & \kappa_{3,h}\,\cong\,\big\uparrow \\
\mathcal{P}_{\rm c}^1(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^0(\mathcal{T}_h) \\
\big\downarrow\;\big\uparrow I_{{\rm grad},h}^1 & & \big\downarrow\;\big\uparrow I_{{\rm curl},h}^1 & & \big\downarrow\;\big\uparrow I_{{\rm div},h}^1 & & \big\downarrow\;\big\uparrow \pi_h^0 \\
\mathcal{P}_{\rm c}^k(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

with κ_h de Rham map, $I_{\bullet,h}^1$ interpolator, and π_h^0 L^2-orthogonal projector
- By the de Rham Theorem, the first two rows have isomorphic cohomologies
- **The two bottom rows fulfill the assumptions of the theorem!**

Shifting point of view I

- Denote by “dofs” the standard FE degrees of freedom
- By unisolvency, we have

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h}\,\cong\uparrow & & \kappa_{1,h}\,\cong\uparrow & & \kappa_{2,h}\,\cong\uparrow & & \kappa_{3,h}\,\cong\uparrow \\
\mathcal{P}_{\mathrm{c}}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^0(\mathcal{T}_h) \\
\mathsf{dofs}^{-1}\,\cong\uparrow & & \mathsf{dofs}^{-1}\,\cong\uparrow & & \mathsf{dofs}^{-1}\,\cong\uparrow & & \mathsf{dofs}^{-1}\,\cong\uparrow \\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

Shifting point of view II

- In the previous diagram, we can erase the middle row
- Set $K_h := \kappa_h \circ \mathrm{dofs}^{-1}$, i.e., for all $(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$
\begin{aligned}
K_{0,h}\underline{q}_h(V) &:= q_V \quad \forall V \in \mathcal{V}_h, & K_{1,h}\underline{v}_h(E) &:= |E| v_E \quad \forall E \in \mathcal{E}_h,\\
K_{2,h}\underline{w}_h(F) &:= |F| v_F \quad \forall F \in \mathcal{F}_h, & K_{3,h} r_h &:= |T| r_T \quad \forall T \in \mathcal{T}_h
\end{aligned}
$$

- These maps induce the following isomorphisms:

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\uparrow {\scriptstyle K_{0,h} \atop \cong} & & \uparrow {\scriptstyle K_{1,h} \atop \cong} & & \uparrow {\scriptstyle K_{2,h} \atop \cong} & & \uparrow {\scriptstyle K_{3,h} \atop \cong} \\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h := K_{1,h}^{-1} \partial_0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h := K_{2,h}^{-1} \partial_1 K_{1,h}, \quad \underline{D}_h^0 \underline{w}_h := K_{3,h}^{-1} \partial_2 K_{2,h},$$

- Notice that, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } \underline{D}_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
{\scriptstyle K_{0,h}}\,\cong \uparrow & & {\scriptstyle K_{1,h}}\,\cong \uparrow & & {\scriptstyle K_{2,h}}\,\cong \uparrow & & {\scriptstyle K_{3,h}}\,\cong \uparrow \\
\mathbb{R}^{\mathcal{V}_h} & \xrightarrow{\underline{G}_h^0} & \mathbb{R}^{\mathcal{E}_h} & \xrightarrow{\underline{C}_h^0} & \mathbb{R}^{\mathcal{F}_h} & \xrightarrow{\underline{D}_h^0} & \mathbb{R}^{\mathcal{T}_h}
\end{array}$$

- **Still true with $\mathcal{M}_h$ CW complex associated to a polyhedral mesh!**

A closer look to the discrete operators

- $\underline{G}_h^0$, $\underline{C}_h^0$, and $\underline{D}_h^0$ are actually the mimetic operators[4]:

$$\underline{G}_h^0 \underline{q}_h := \left(G_E^0 \underline{q}_E = \frac{q_{V_2} - q_{V_1}}{|E|} \right)_{E \in \mathcal{E}_h},$$

$$\underline{C}_h^0 \underline{v}_h := \left(C_F^0 \underline{v}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} |E| v_E \right)_{F \in \mathcal{F}_h},$$

$$\underline{D}_h^0 \underline{w}_h := \left(D_T^0 \underline{w}_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \omega_{TF} |F| w_F \right)_{T \in \mathcal{T}_h}$$

- These operators are polynomially exact

[4] See, e.g., [Beirão da Veiga et al., 2014] and [Bonelle and Ern, 2014]

The arbitrary-order case $k \geq 0$

$$\underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathrm{curl},h} \xrightarrow{\underline{C}^k_h} \underline{X}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h)$$

	V	E	F	T
$\underline{X}^k_{\mathrm{grad},h}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}^k_{\mathrm{curl},h}$	–	$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{X}^k_{\mathrm{div},h}$	–	–	$\mathcal{P}^k(F)$	$\mathcal{N}^k(T)$
$\underline{X}^k_{\mathrm{L},h}$	–	–	–	$\mathcal{P}^k(T)$

- Discrete de Rham (DDR) [DP et al., 2020, DP and Droniou, 2023a]
- Serendipity version [DP and Droniou, 2023b][5]
- Extension to other complexes and applications in various subsequent papers

[5]See [Beirão da Veiga et al., 2018b] for a preliminary work on this subject

An example: The arbitrary order curl space I

- The face curl $C_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.

$$\int_F C_F^k \underline{v}_F \, q = \int_F v_F \cdot \mathrm{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \, q \quad \forall q \in \mathcal{P}^k(F)$$

- The tangent trace $\gamma_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)^2$ is s.t., $\forall (r,w) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$,

$$\int_F \gamma_F^k \underline{v}_F \cdot (\mathrm{rot}_F \, r + w) = \int_F C_F^k \underline{v}_F \, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \, r + \int_F v_F \cdot w$$

- From γ_F^k, we build the element curl $C_T^k : \underline{X}_{\mathrm{curl},h}^k \to \mathcal{P}^k(T)^3$ similarly to C_F^k and set

$$\begin{aligned} \underline{C}_h^k : \underline{X}_{\mathrm{curl},h}^k &\to \underline{X}_{\mathrm{div},h}^k \\ \underline{v}_h &\mapsto \left((\pi_{\mathcal{N}^k(T)} C_T^k \underline{v}_T)_{T \in \mathcal{T}_h}, (C_F^k \underline{v}_F)_{F \in \mathcal{F}_h} \right) \end{aligned}$$

An example: The arbitrary order curl space II

- From C_T^k and γ_F^k, we build an element potential $P^k_{\text{curl},T} : \underline{X}^k_{\text{curl},T} \to \mathcal{P}^k(T)^3$
- The local L^2-product in $\underline{X}^k_{\text{curl},T}$ is

$$(\underline{w}_T, \underline{v}_T)_{\text{curl},T} := \int_T P^k_{\text{curl},T}\underline{w}_T \cdot P^k_{\text{curl},T}\underline{v}_T + \text{stab.}$$

  where stab. penalizes $\underline{v}_T - \underline{I}^k_{\text{curl},T} P^k_{\text{curl},T}\underline{v}_T$ in a least-square sense
- The global discrete L^2-product is obtained assemblying element-wise:

$$(\underline{w}_h, \underline{v}_h)_{\text{curl},h} := \sum_{T \in \mathcal{T}_h} (\underline{w}_T, \underline{v}_T)_{\text{curl},T}$$

An example of numerical scheme

- Let us consider again the magnetostatics problem:
  Find $(H, A) \in H(\mathrm{curl}; \Omega) \times H(\mathrm{div}; \Omega)$ s.t.

$$\begin{aligned} \mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\, \tau &= 0 && \forall \tau \in H(\mathrm{curl}; \Omega), \\ \int_\Omega \mathrm{curl}\, H \cdot v + \int_\Omega \mathrm{div}\, A\, \mathrm{div}\, v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div}; \Omega) \end{aligned}$$

- A DDR scheme for this problem is obtained with obvious substitutions:
  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}^k_{\mathrm{curl},h} \times \underline{X}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned} \mu(\underline{H}_h, \underline{\tau}_h)_{\mathrm{curl},h} - (\underline{A}_h, \underline{C}^k_h \underline{\tau}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\tau}_h \in \underline{X}^k_{\mathrm{curl},h}, \\ (\underline{C}^k_h \underline{H}_h, \underline{v}_h)_{\mathrm{div},h} + (\underline{D}^k_h \underline{A}_h, \underline{D}^k_h \underline{v}_h)_{\mathrm{L},h} &= (\underline{I}^k_{\mathrm{div},h} J, \underline{v}_h)_{\mathrm{div},h} && \forall \underline{v}_h \in \underline{X}^k_{\mathrm{div},h} \end{aligned}$$

- **Stability follows mimicking the continuous argument for well-posedness**

Restoring function spaces

- The above construction is fully discrete. Can we restore function spaces?

$$
\begin{array}{lccccccc}
 & H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{VEM} & V^k_{\text{grad},h} & \xrightarrow{\text{grad}} & V^k_{\text{curl},h} & \xrightarrow{\text{curl}} & V^k_{\text{div},h} & \xrightarrow{\text{div}} & \mathcal{P}^k(\mathcal{T}_h) \\
 & \text{dofs}^{-1} \cong \uparrow & & \text{dofs}^{-1} \cong \uparrow & & \text{dofs}^{-1} \cong \uparrow & & \updownarrow \\
\text{DDR} & \underline{X}^k_{\text{grad},h} & \xrightarrow{\underline{G}^k_h} & \underline{X}^k_{\text{curl},h} & \xrightarrow{\underline{C}^k_h} & \underline{X}^k_{\text{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h)
\end{array}
$$

- The spaces $V^k_{\bullet,h}$ are finite-dimensional but not polynomial in general
- This is the key idea of the Virtual Element Method (VEM)[6]

[6] See [Beirão da Veiga et al., 2013] and [Beirão da Veiga et al., 2016 and 2018a] for complexes, [Beirão da Veiga, Dassi, DP, and Droniou, 2022] for the present construction

An example: The virtual curl space I

- For $X \in \mathcal{T}_h \cup \mathcal{F}_h$, let $\mathcal{P}^{k-1|k+1}(X)$ be s.t. $\mathcal{P}^{k+1}(X) = \mathcal{P}^{k-1}(X) \oplus \mathcal{P}^{k-1|k+1}(X)$
- The curl space on a face $F \in \mathcal{F}_h$ is

$$\begin{aligned} V^k_{\text{curl}}(F) := \Big\{ & v \in L^2(F)^{d-1} \;:\; \operatorname{div}_F v \in \mathcal{P}^{k+1}(F),\ \operatorname{rot}_F v \in \mathcal{P}^k(F), \\ & v \cdot t_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_F, \\ & \int_F (v - \pi_{\mathcal{P}^k(F)} v) \cdot (x - x_F)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(F) \Big\} \end{aligned}$$

- The curl space on a mesh element $T \in \mathcal{T}_h$ is

$$\begin{aligned} V^k_{\text{curl}}(T) := \Big\{ & v \in L^2(T)^d \;:\; n_{TF} \times (v \times n_{TF}) \in V^k_{\text{curl}}(F) \text{ for all } F \in \mathcal{F}_T, \\ & \int_T (\operatorname{curl} v - \pi_{\mathcal{P}^k(T)^d} \operatorname{curl} v) \cdot (x_T \times w) = 0 \text{ for all } w \in \mathcal{P}^{k-1|k}(T)^d, \\ & \int_T (v - \pi_{\mathcal{P}^k(T)^d} v) \cdot (x - x_T)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(T) \Big\} \end{aligned}$$

An example: The virtual curl space II

- The global curl space is defined setting

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) := \left\{ v \in H(\mathrm{curl};\Omega) \;:\; v_{|T} \in V^k_{\mathrm{curl}}(T) \text{ for all } T \in \mathcal{T}_h \right\}$$

- The degrees of freedom are:
  - For each edge $E \in \mathcal{E}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_E (v \cdot t_E)\, p \in \mathbb{R} \quad \forall p \in \mathcal{P}^k(E)$$

  - If $k \geq 1$, for each face $F \in \mathcal{F}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_F n_F \times (v \times n_F) \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(F)$$

  - If $k \geq 1$, for each element $T \in \mathcal{T}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_T v \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(T)$$

Extension to differential forms I

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a polytopal domain or manifold
- FE have been extended to the de Rham complex of differential forms[7]:

$$\cdots \longrightarrow H\Lambda^{i-1}(\Omega) \xrightarrow{\mathrm{d}^i} H\Lambda^{i}(\Omega) \longrightarrow \cdots$$

- This has lead to new elements, advanced complexes, etc.
- A Polytopal Exterior Calculus (PEC) framework has been recently presented in [Bonaldi, DP, Droniou and Hu, 2023]

[7] See, e.g., [Bossavit, 1988, Hiptmair, 2002, Arnold et al., 2006]

Outline

Examples of research avenues on polytopal complexes

- Generalization to PEC of relevant analytical results:
  - Poincaré- and Sobolev-type inequalities
  - Adjoint consistency of discrete differential operators
  - ...
- Extension to advanced complexes (e.g., through the BGG construction[8])
  - Stokes complex [Beirão da Veiga et al., 2020, Hanot, 2023]
  - Two-dimensional rot-rot complex [DP, 2023]
  - Three-dimensional div-div complex [DP and Hanot, 2024]
  - ...
- Extension of serendipity techniques to PEC
- Applications of the new complexes to model problems
- **See the other talks of this session for more!**

[8] [Arnold and Hu, 2021]

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