Fully discrete polynomial de Rham sequences of arbitrary degree on polyhedral meshes

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References and acknowledgements

- Local DDR sequences [DP, Droniou and Rapetti, 2020]
- Global DDR sequences and stability [DP and Droniou, 2020a]
- Primal and dual consistency [DP and Droniou, ongoing]
- See [DP and Droniou, 2020b] for polytopal analysis tools

Outline

1. Introduction and motivation
2. Discrete de Rham (DDR) sequences
3. Application to magnetostatics

A (not so simple) model problem I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain that does not enclose any void
- Let a current density $\boldsymbol{f} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$ be given
- We consider the problem: Find the magnetic field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^3$ and the vector potential $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\boldsymbol{\sigma} - \mathbf{curl}\, \boldsymbol{u} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\, \boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{u} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- The extension to variable magnetic permeability is straightforward

A (not so simple) model problem II

- In weak formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{\tau} &= 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} & \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- Well-posedness hinges on the exactness of the following portion of the de Rham sequence:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathrm{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- **This exactness property is also needed at the discrete level!**

The Finite Element way

Local spaces

- **Key idea:** define subspaces that form exact sequence
- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree} \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by $\boldsymbol{x}_T$ the barycenter of T,

$$\begin{aligned}
\mathcal{P}^k(T)^3 &= \underbrace{\mathbf{grad}\,\mathcal{P}^{k+1}(T)}_{\mathcal{G}^k(T)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}^{k-1}(T)^3}_{\mathcal{G}^{\mathrm{c},k}(T)} \\
&= \underbrace{\mathbf{curl}\,\mathcal{P}^{k+1}(T)^3}_{\mathcal{R}^k(T)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}^{k-1}(T)}_{\mathcal{R}^{\mathrm{c},k}(T)}
\end{aligned}$$

- Define the trimmed spaces

$$\mathcal{N}^k(T) \coloneqq \mathcal{G}^k(T) \oplus \mathcal{G}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\mathcal{RT}^k(T) \coloneqq \mathcal{R}^k(T) \oplus \mathcal{R}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

The Finite Element way

Global FE sequence

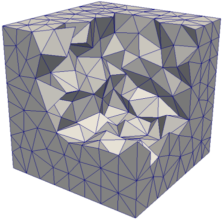


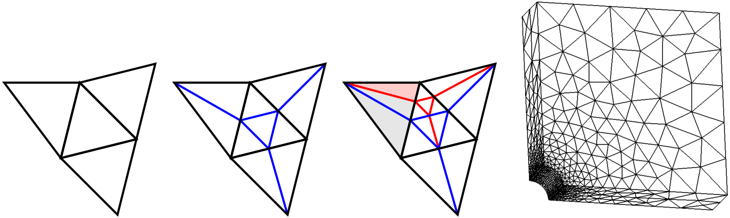
Figure: Conforming tetrahedral mesh of the unit cube (clip)

- Let $\mathcal{T}_h = \{T\}$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form the global FE sequence

$$\mathbb{R} \xrightarrow{i_\Omega} \mathcal{P}_c^{k+1}(\mathcal{T}_h) \xrightarrow{\textbf{grad}} \boldsymbol{N}^k(\mathcal{T}_h) \xrightarrow{\textbf{curl}} \boldsymbol{\mathcal{RT}}^k(\mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- **This procedure only works on conforming meshes!**

The Finite Element way
Shortcomings



- Approach limited to conforming meshes with standard elements
- $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
- $\Longrightarrow$ complex geometries may require a large number of elements
- $\Longrightarrow$ the element shape cannot be adapted to the solution
- The implementation of high-order versions may be tricky
- ...

The discrete de Rham (DDR) approach I

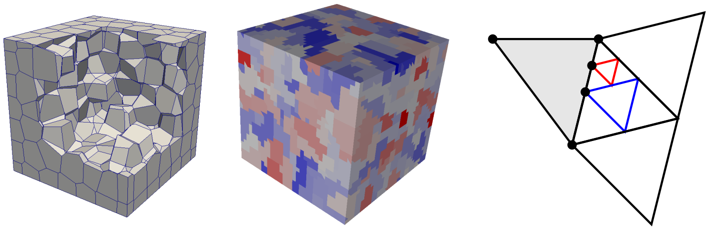


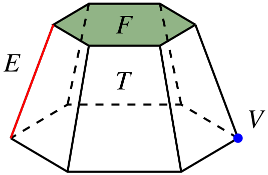
Figure: Examples of polytopal meshes supported by the DDR approach

- **Key idea:** replace spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{G}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of general polyhedral meshes and high-order **(!)**
- Exactness proved at the discrete level (directly usable for stability)
- (Relatively) simple implementation of high-order versions

The discrete de Rham (DDR) approach II



- The fully discrete spaces are spanned by vectors of polynomials
- Polynomial components attached to geometric objects to mimic
  - full continuity for the approximation of $H^1(\Omega)$
  - continuity of tangential traces for the approximation of $\boldsymbol{H}(\mathbf{curl};\Omega)$
  - continuity of normal traces for the approximation of $\boldsymbol{H}(\mathrm{div};\Omega)$
- Selected so as to enable the reconstruction of consistent
  - discrete vector calculus operators
  - (scalar or vector) discrete potentials

Outline

- Let F be a simply connected polygon embedded in $\mathbb{R}^3$
- Let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q \coloneqq \varrho_{-\pi/2}(\mathbf{grad}_F\, q) \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneqq \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v})$$

- We derive a discrete counterpart of the exact local sequence:

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

$$\begin{aligned}\mathcal{P}^k(F)^2 &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{G}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp \mathcal{P}^{k-1}(F)}_{\mathcal{G}^{\mathrm{c},k}(F)}\end{aligned}$$

The two-dimensional case

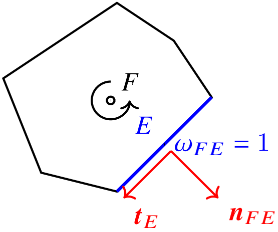
A key remark

- Denote by $\pi_{\mathcal{P},F}^{k-1}$ the L^2-orthogonal projector on $\mathcal{P}^{k-1}(F)$
- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

$$= -\int_F \pi_{\mathcal{P},F}^{k-1} q\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$



The two-dimensional case

Discrete $H^1(F)$ space

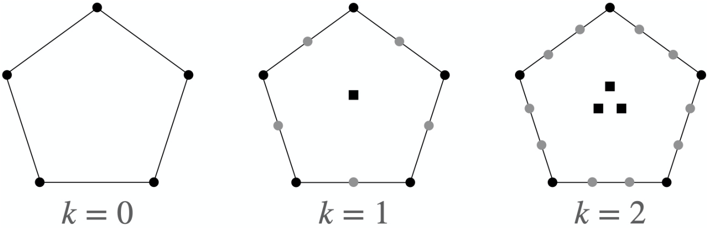


Figure: Number of degrees of freedom for $\underline{X}^k_{\mathrm{grad},F}$ for $k \in \{0,1,2\}$

- The discrete counterpart of $H^1(F)$ is

$$\underline{X}^k_{\mathrm{grad},F} := \left\{ \underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_F) \right\}$$

- The interpolator $\underline{I}^k_{\mathrm{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathrm{grad},F}$ is s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathrm{grad},F} q := (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \;\forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \;\forall V \in \mathcal{V}_F$$

The two-dimensional case

Reconstructions in $\underline{X}^k_{\mathbf{grad},F}$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- We reconstruct similarly a face potential in $\mathcal{P}^{k+1}(F)$

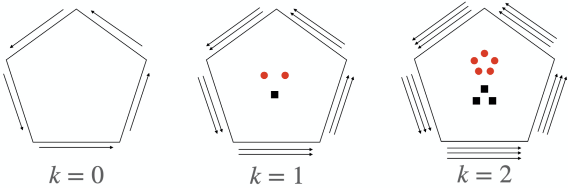


Figure: Number of degrees of freedom for $\underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ for $k \in \{0,1,2\}$

- We reason starting from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^k(F) := \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$,

$$\int_F \mathrm{rot}_F\,\boldsymbol{v}\; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) \underbrace{q_{|E}}_{\in \mathcal{P}^k(E)} \qquad \forall q \in \mathcal{P}^k(F)$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot};F)$:

$$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}^{\mathrm{c}}_{\boldsymbol{\mathcal{R}},F}, (v_E)_{E \in \mathcal{E}_F}\big) : \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}^{\mathrm{c}}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \ q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \ q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \ q \quad \forall q \in \mathcal{P}^k(F)$$

- Define the interpolator $\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} : H^1(F)^2 \to \underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ s.t., $\forall \boldsymbol{v} \in H^1(F)^2$,

$$\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v} := \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F} \boldsymbol{v}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F} \boldsymbol{v}, \left(\pi^k_{\mathcal{P},E}(\boldsymbol{v}_{|E} \cdot \boldsymbol{t}_E)\right)_{E \in \mathcal{E}_F}\right).$$

- C_F^k is polynomially consistent by construction:

$$C_F^k\left(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v}\right) = \mathrm{rot}_F \ \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^k(F)$$

- By similar principles, we reconstruct a vector potential in $\mathcal{P}^k(F)^2$

Theorem (Exactness of the two-dimensional local DDR sequence)

If F is simply connected, the following local sequence is exact:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\},$$

where $\underline{\boldsymbol{G}}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ *is the discrete gradient s.t.,* $\forall \underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F := \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), (G^k_E \underline{q}_F)_{E \in \mathcal{E}_F} \right)$$

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (polygon)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

Table: Polynomial components for the two-dimensional spaces

- Interpolators = component-wise L^2-projections
- Discrete operators = L^2-projections of full operator reconstructions

The three-dimensional case I

Exact sequence

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\text{grad},T}} \underline{X}^k_{\text{grad},T} \xrightarrow{\underline{G}^k_T} \underline{\boldsymbol{X}}^k_{\text{curl},T} \xrightarrow{\underline{C}^k_T} \underline{\boldsymbol{X}}^k_{\text{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F (face)	T (polyhedron)
$\underline{X}^k_{\text{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\text{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\text{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\text{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\text{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\text{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Table: Polynomial components for the three-dimensional spaces

Theorem (Exactness of the three-dimensional local DDR sequence)

If the polyhedron T has a trivial topology, this sequence is exact.

Lemma (Commutative diagram with the sequence of trimmed spaces)

The following commutative diagram holds, expressing the polynomial consistency of the discrete vector calculus operators:

$$
\begin{array}{ccccccc}
\mathcal{P}^{k+1}(T) & \xrightarrow{\textbf{grad}} & \boldsymbol{\mathcal{N}}^k(T) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{RT}}^k(T) & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(T) \\
\downarrow \underline{I}^k_{\textbf{grad},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\textbf{curl},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},T} & & \downarrow i_T \\
\underline{X}^k_{\textbf{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\textbf{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T)
\end{array}
$$

- Emulating integration by part formulas, define the local potentials

$$\begin{aligned} P^{k+1}_{\mathbf{grad},T} &: \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T), \\ \boldsymbol{P}^{k}_{\mathbf{curl},T} &: \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^{k}(T)^3, \\ \boldsymbol{P}^{k}_{\mathrm{div},T} &: \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \mathcal{P}^{k}(T)^3 \end{aligned}$$

- Based on these potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are polynomially exact

- Let $\Omega \subset \mathbb{R}^3$ as before and let $\mathcal{T}_h$ be a polyhedral mesh
- Global DDR spaces are defined gluing boundary components:

$$\underline{X}_{\mathbf{grad},h}^k, \qquad \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k, \qquad \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}_h^k, \qquad \underline{\boldsymbol{C}}_h^k, \qquad D_h^k$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise
- The global DDR sequence is

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},h}^k} \underline{X}_{\mathbf{grad},h}^k \xrightarrow{\underline{\boldsymbol{G}}_h^k} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \xrightarrow{\underline{\boldsymbol{C}}_h^k} \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Outline

- Continuous problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- The global bilinear forms are approximated substituting

$$\begin{aligned} (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} &\leftarrow \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \\ (\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} &\leftarrow \int_\Omega \mathbf{curl}\,\boldsymbol{\tau}\cdot\boldsymbol{v} \\ \int_\Omega D_h^k\underline{\boldsymbol{w}}_h\ D_h^k\underline{\boldsymbol{v}}_h &\leftarrow \int_\Omega \mathrm{div}\,\boldsymbol{w}\ \ \mathrm{div}\,\boldsymbol{v} \end{aligned}$$

- The current density linear form is l_h, defined similarly

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h \, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}
\end{aligned}
$$

- Stability hinges on the exactness of the portion

$$
\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}
$$

A DDR scheme for magnetostatics

Global exactness I

Theorem (Exactness properties of the global DDR sequence)

Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain. Then, it holds

$$\operatorname{Im} D_h^k = \mathcal{P}^k(\mathcal{T}_h).$$

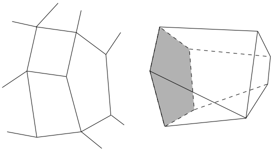
If Ω does not enclose any void, we additionally have

$$\operatorname{Im} \underline{\boldsymbol{C}}_h^k = \operatorname{Ker} D_h^k.$$

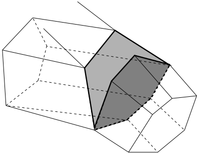
- $\operatorname{Im} D_h^k = \mathcal{P}^k(\mathcal{T}_h)$ follows from the classical Fortin's argument
- The inclusion $\operatorname{Im} \underline{\boldsymbol{C}}_h^k \subset \operatorname{Ker} D_h^k$ results from local exactness
- We prove $\operatorname{Ker} D_h^k \subset \operatorname{Im} \underline{\boldsymbol{C}}_h^k$ in two steps. Let $\underline{\boldsymbol{v}}_h \in \operatorname{Ker} D_h^k$. Then:
  - Local exactness gives $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$ s.t. $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{\tau}}_T$ for all $T \in \mathcal{T}_h$
  - The local vectors are then glued together

To glue together local vectors, we use the fact that the mesh can be topologically assembled by a succession of the following operations:

1. Add a new element by gluing one ot its faces to an element in the mesh



2. Glue together two faces of elements in the mesh s.t. the edges along which the faces are already glued together form a connected path



This is only possible since Ω does not enclose any void!

Theorem (Well-posedness)

Let $\Omega \subset \mathbb{R}^3$ be an open simply connected polyhedral domain that does not enclose any void. Then, $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ is unique and there exists $C > 0$ independent of h s.t.

$$\|\underline{\boldsymbol{\sigma}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h\|_{\mathrm{div},h} + \|\underline{\boldsymbol{u}}_h\|_{\mathrm{div},h} + \|D^k_h \underline{\boldsymbol{u}}_h\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_\Omega.$$

Proof.

Analogous to the continuous case since all the relevant properties have been reproduced at the discrete level. □

- Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a regular polyhedral mesh sequence
- We consider a known solution $(\boldsymbol{\sigma}, \boldsymbol{u})$ to assess convergence rate r s.t.

$$\text{approximation error} \propto h^r$$

- The error

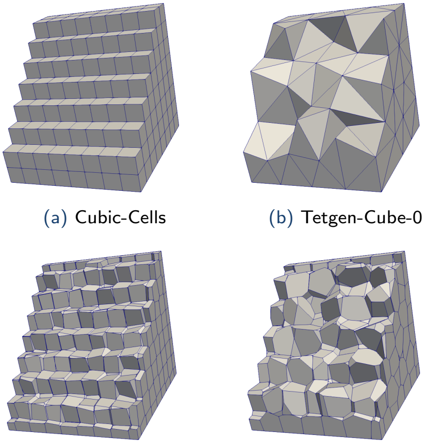
$$(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h) \coloneqq (\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u})$$

  is measured in the natural energy norm s.t.

$$\|(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h)\|_{\mathrm{en},h} \coloneqq \left[(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{e}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{\varepsilon}}_h)_{\mathrm{div},h}\right]^{\frac{1}{2}}$$

- The implementation is based on the HArDCore3D C++ library[1]

[1] See `https://tinyurl.com/HarDCore3D`



(a) Cubic-Cells (b) Tetgen-Cube-0

(c) Voro-small-0 (d) Voro-small-1

Figure: Mesh families used in the numerical tests

Numerical examples

Convergence in the energy norm

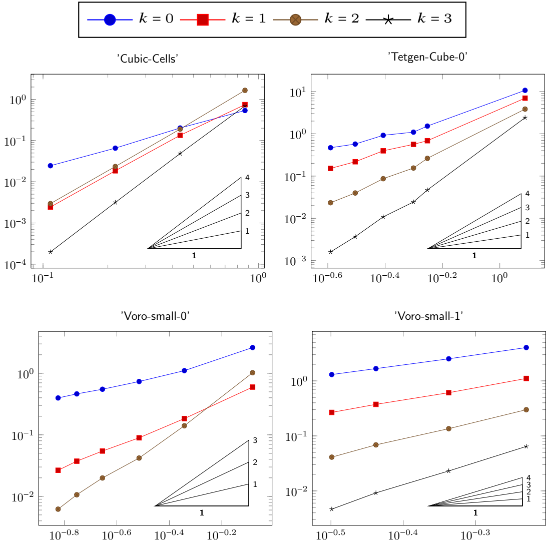


Figure: Energy error versus mesh size h. We have $\|(\underline{e}_h, \underline{\varepsilon}_h)\|_{\mathrm{en},h} \propto h^{k+1}$

Conclusions and perspectives

- A novel approach for the numerical solution of PDE problems
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to variable coefficients and nonlinearities

- Applications (electromagnetism, incompressible fluid mechanics,. . .)
- Formalization using differential forms (ongoing)
- Development of novel sequences (e.g., elasticity)
- . . .

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