Non-standard applications of the Raviart–Thomas–Nédélec element

A HHO method for the Brinkman problem robust in the Darcy and Stokes limits

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The Raviart–Thomas–Nédélec finite element I

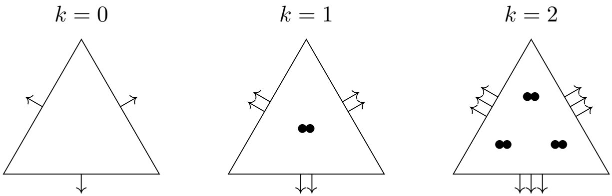

Figure: Degrees of freedom for $\mathbb{RTN}^k(T)$

- Let $d \geq 1$, T denote a d-simplex, and $k \geq 0$
- We consider here the Raviart–Thomas–Nédélec space

$$\mathbb{RTN}^k(T) := \mathbb{P}^k(T)^d + \boldsymbol{x}\mathbb{P}^k(T)$$

- A function $\boldsymbol{v} \in \mathbb{RTN}^k(T)$ is uniquely defined by the quantities

$$\left\{(\boldsymbol{v}, \boldsymbol{w})_T \;:\; \boldsymbol{w} \in \mathbb{P}^{k-1}(T)^d\right\} \text{ and } \left\{(\boldsymbol{v}\cdot\boldsymbol{n}_{TF}, q)_F \;:\; q \in \mathbb{P}^k(F)\right\}$$

The Raviart–Thomas–Nédélec finite element II

- Introduced in [Raviart and Thomas, 1977, Nédélec, 1980]
- Tailored to mixed Darcy: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} \nu\boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{0} && \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= g && \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\ \int_\Omega p &= 0 \end{aligned}$$

- We show new applications of this finite element:
  - Robust HHO method for Brinkman [Botti, DP, Droniou, 18]
  - (Stable gradient reconstruction for HHO: see [DP et al., 2018])

The Brinkman problem

- Let $\mu : \Omega \to \mathbb{R}$ and $\nu : \Omega \to \mathbb{R}$ be piecewise constant and s.t.

$$0 < \underline{\mu} \leq \mu \leq \overline{\mu}, \qquad 0 \leq \underline{\nu} \leq \nu \leq \overline{\nu}$$

- The Brinkman problem reads: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(2\mu\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \nu\boldsymbol{u} + \boldsymbol{\nabla}p &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= g && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \\ \int_\Omega p &= 0 \end{aligned}$$

- It locally behaves like a Stokes or a Darcy problem (singular limit)
- **Goal: Identify the local regime and handle all regimes robustly**

State of the art

- Naïve choices are not uniformly well-behaved [Mardal et al., 2002]:
  - Crouzeix–Raviart fails to converge in the Darcy limit
  - Taylor–Hood and the minielement experience convergence losses
- Several fixes proposed, including:
  - Low-order stabilised FE [Burman and Hansbo, 2007]
  - Stabilised equal-order FE [Braack and Schieweck, 2011]
  - Generalisation of the minielement [Juntunen and Stenberg, 2010]
  - Stabilised $H(\mathrm{div}; \Omega)$-conforming FE [Könnö and Stenberg, 2011]
  - 2d $H(\mathrm{div}; \Omega)$-conforming VEM [Vacca, 2018]

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  - 2d $H(\mathrm{div};\Omega)$-conforming VEM [Vacca, 2018]
- Recurrent problems:
  - Darcy and Stokes contributions are not equilibrated
  - Local regimes not clearly identified

In a nutshell I

Key idea [Botti, DP, Droniou, 2018]: Replace FE by HHO

Features of HHO methods:

- Construction valid for arbitrary space dimensions
- Arbitrary approximation order
- Robustness with respect to the variations of the physical coefficients
- Reduced computational cost after static condensation
- (Capability of handling general polyhedral meshes)
- New schemes even on standard meshes

In a nutshell II

- Hybrid velocity, piecewise polynomial pressure
  - Inf-sup stable for arbitrary polynomial degree
  - Possibility to statically condense a large subset of the unknowns
- Local Stokes velocity reconstruction in $\mathbb{P}^{k+1}(T)^d$
  - Gain of (up to) two orders w.r. to element unknowns
  - Tailored to the Stokes regime
- Local Darcy velocity reconstruction in $\mathbb{RTN}^k(T)$
  - Equilibrated Stokes-Darcy terms in $\mathcal{O}(h^{k+1})$
  - Tailored to the Darcy regime

Projectors on local polynomial spaces I

- Let $\mathcal{T}_h$ denote a polytopal mesh with faces collected in $\mathcal{F}_h$
- HHO methods hinge on projectors on local polynomial spaces
- With X element or face, the L^2-projector $\pi_X^\ell : L^1(T) \to \mathbb{P}^\ell(X)$ is s.t.

$$(\pi_X^\ell v - v, w)_X = 0 \text{ for all } w \in \mathbb{P}^\ell(X)$$

- For $T \in \mathcal{T}_h$, the strain projector $\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^l : H^1(T)^d \to \mathbb{P}^\ell(T)^d$ is s.t.

$$(\boldsymbol{\nabla}_{\rm s}(\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^\ell \boldsymbol{v} - \boldsymbol{v}), \boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^\ell(T)^d$$

and

$$\int_T \boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^\ell \boldsymbol{v} = \int_T \boldsymbol{v}, \qquad \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^\ell \boldsymbol{v} = \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{v}$$

Projectors on local polynomial spaces II

Theorem (Approximation properties of the strain projector)

Assume T star-shaped with respect to every point of a ball of radius $\geq \varrho h_T$. Let two integers $\ell \geq 1$ and $s \in \{1, \ldots, \ell+1\}$ be given. Then, it holds with hidden constant depending only on d, ϱ, ℓ, and s such that, for all $m \in \{0, \ldots, s\}$ and all $v \in H^s(T)^d$,

$$|\boldsymbol{v} - \boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{\ell}\boldsymbol{v}|_{H^m(T)^d} \lesssim h_T^{s-m}|\boldsymbol{v}|_{H^s(T)^d}.$$

and

$$|\boldsymbol{v} - \boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{\ell}\boldsymbol{v}|_{H^m(\mathcal{F}_T)^d} \lesssim h_T^{s-m-\frac{1}{2}}|\boldsymbol{v}|_{H^s(T)^d}$$

with $H^m(\mathcal{F}_T)$ broken Sobolev space on $\mathcal{F}_T$.

Proof.

See [Appendix A.2, Botti, DP, Droniou, 2018]. □

Computing $\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}$ from L^2-projections of degree k

- For all $\boldsymbol{v} \in H^1(T)^d$ and all $\boldsymbol{w} \in C^\infty(\overline{T})^d$, it holds that

$$(\nabla_{\rm s}\boldsymbol{v}, \nabla_{\rm s}\boldsymbol{w})_T = -(\boldsymbol{v}, \nabla\cdot\nabla_{\rm s}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{v}, \nabla_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF})_F$$

- For $k \geq 0$ and $l := \max\{0, k-1\}$, letting $\boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d$, we get

$$(\nabla_{\rm s}\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}\boldsymbol{v}, \nabla_{\rm s}\boldsymbol{w})_T = -(\boldsymbol{\pi}_T^l\boldsymbol{v}, \nabla\cdot\nabla_{\rm s}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{\pi}_F^k\boldsymbol{v}_{|F}, \nabla_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF})_F$$

- Moreover, it can be easily seen that

$$\int_T \boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}\boldsymbol{v} = \int_T \boldsymbol{\pi}_T^l\boldsymbol{v}, \quad \int_T \nabla_{\rm ss}\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}\boldsymbol{v} = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F \left(\boldsymbol{n}_{TF}\otimes\boldsymbol{\pi}_F^k\boldsymbol{v} - \boldsymbol{\pi}_F^k\boldsymbol{v}\otimes\boldsymbol{n}_{TF}\right)$$

- **Hence, $\boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}\boldsymbol{v}$ can be computed from $\boldsymbol{\pi}_T^l\boldsymbol{v}$ and $(\boldsymbol{\pi}_F^k\boldsymbol{v}_{|F})_{F\in\mathcal{F}_T}$!**

Local space of discrete velocity unknowns

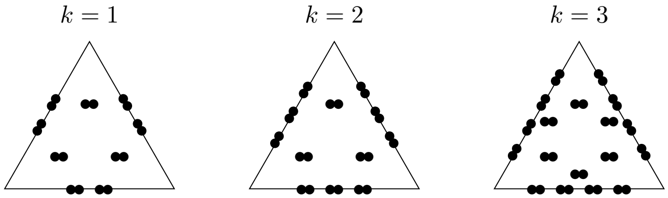

Figure: Degrees of freedom for $\underline{\boldsymbol{U}}_T^k$ for $k \in \{1, 2\}$

- Assume $\mathcal{T}_h$ matching simplicial, let $k \geq 1$, and set $l := \max\{k-1, 1\}$
- For all $T \in \mathcal{T}_h$, we define the local space of velocity unknowns

$$\underline{\boldsymbol{U}}_T^k := \left\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \,:\, \boldsymbol{v}_T \in \mathbb{P}^l(T)^d \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{\boldsymbol{I}}_T^k : H^1(T)^d \to \underline{\boldsymbol{U}}_T^k$ is s.t., for all $\boldsymbol{v} \in H^1(T)^d$,

$$\underline{\boldsymbol{I}}_T^k \boldsymbol{v} := (\boldsymbol{\pi}_T^l \boldsymbol{v}, (\pi_F^k \boldsymbol{v}_{|F})_{F \in \mathcal{F}_T})$$

A high-order Stokes velocity reconstruction

- Let $T \in \mathcal{T}_h$. We define the local Stokes velocity reconstruction

$$\boxed{\boldsymbol{r}_{\mathrm{S},T}^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)^d}$$

s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$ and all $\boldsymbol{w} \in \mathbb{P}^{k+1}(T)^d$,

$$(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{v}}_T, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w})_T = -(\boldsymbol{v}_T, \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w})_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{v}_F, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w} \boldsymbol{n}_{TF})_F$$

and

$$\int_T \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \quad \int_T \boldsymbol{\nabla}_{\mathrm{ss}} \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{v}}_T = \frac{1}{2} \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{n}_{TF} \otimes \boldsymbol{v}_F - \boldsymbol{v}_F \otimes \boldsymbol{n}_{TF})$$

- By construction, we have

$$\boxed{\boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{I}}_T^k = \boldsymbol{\pi}_{\boldsymbol{\varepsilon},T}^{k+1}}$$

- $\boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{I}}_T^k$ has therefore optimal approximation properties in $\mathbb{P}^{k+1}(T)^d$

Global spaces

- Local spaces are glued by enforcing single-valuedness at interfaces:

$$\underline{\boldsymbol{U}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^l(T)^d \quad \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_h \Big\}$$

- Boundary conditions are strongly incorporated in the subspace

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The pressure is sought in the broken polynomial space

$$P_h^k := \left\{ q_h \in \mathbb{P}^k(\mathcal{T}_h) \ : \ \int_\Omega q_h = 0 \right\}$$

Stokes term

- Inside $T \in \mathcal{T}_h$, we approximate the Stokes term with

$$\boxed{\mathrm{a}_{\mathrm{S},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := (2\mu_T \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{u}}_T, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{v}}_T)_T + \mathrm{s}_{\mathrm{S},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)}$$

- The Stokes stabilisation bilinear form is s.t.

$$\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := (2\mu_T)(\boldsymbol{\delta}_{\mathrm{S},T}^l \underline{\boldsymbol{u}}_T, \boldsymbol{\delta}_{\mathrm{S},T}^l \underline{\boldsymbol{v}}_T)_T + \sum_{F \in \mathcal{F}_T} \frac{2\mu_T}{h_F} (\boldsymbol{\delta}_{\mathrm{S},TF}^k \underline{\boldsymbol{u}}_T, \boldsymbol{\delta}_{\mathrm{S},TF}^k \underline{\boldsymbol{v}}_T)_F$$

with Stokes difference operators s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$(\boldsymbol{\delta}_{\mathrm{S},T}^l \underline{\boldsymbol{v}}_T, (\boldsymbol{\delta}_{\mathrm{S},TF}^k \underline{\boldsymbol{v}}_T)_{F \in \mathcal{F}_T}) := \underline{\boldsymbol{I}}_T^k \boldsymbol{r}_{\mathrm{S},T}^{k+1} \underline{\boldsymbol{v}}_T - \underline{\boldsymbol{v}}_T$$

- The global Stokes bilinear form is assembled element-wise:

$$\boxed{\mathrm{a}_{\mathrm{S},h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_{\mathrm{S},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)}$$

A Darcy velocity reconstruction in $\mathbb{RTN}^k(T)$

- The local Darcy velocity reconstruction

$$\boxed{\boldsymbol{r}_{\mathrm{D},T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{RTN}^k(T)}$$

is s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$(\boldsymbol{r}_{\mathrm{D},T}^k \underline{\boldsymbol{v}}_T, \boldsymbol{w})_T = (\boldsymbol{v}_T, \boldsymbol{w})_T \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k-1}(T)^d$$
$$(\boldsymbol{r}_{\mathrm{D},T}^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_{TF}, q)_F = (\boldsymbol{v}_F \cdot \boldsymbol{n}_{TF}, q)_F \qquad \forall F \in \mathcal{F}_T,\ \forall q \in \mathbb{P}^k(F).$$

- A direct verification shows that

$$\boxed{\boldsymbol{r}_{\mathrm{D},T}^k \underline{\boldsymbol{I}}_T^k = \boldsymbol{I}_{\mathbb{RTN},T}^k}$$

where $\boldsymbol{I}_{\mathbb{RTN},T}^k$ is the standard interpolator on $\mathbb{RTN}^k(T)$

Darcy term

- Inside $T \in \mathcal{T}_h$, we approximate the Darcy term with

$$\boxed{\mathrm{a}_{\mathrm{D},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \nu_T(\boldsymbol{r}^k_{\mathrm{D},T}\underline{\boldsymbol{u}}_T, \boldsymbol{r}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T)_T + \mathrm{s}_{\mathrm{D},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)}$$

- The Darcy stabilisation bilinear form is s.t.

$$\mathrm{s}_{\mathrm{D},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \nu_T(\boldsymbol{\delta}^l_{\mathrm{D},T}\underline{\boldsymbol{u}}_T, \boldsymbol{\delta}^l_{\mathrm{D},T}\underline{\boldsymbol{v}}_T)_T + \sum_{F \in \mathcal{F}^{\mathrm{i}}_T} \nu_T h_F(\boldsymbol{\delta}^k_{\mathrm{D},TF}\underline{\boldsymbol{u}}_T, \boldsymbol{\delta}^k_{\mathrm{D},TF}\underline{\boldsymbol{v}}_T)_F$$

with Darcy difference operators s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}^k_T$,

$$(\boldsymbol{\delta}^l_{\mathrm{D},T}\underline{\boldsymbol{v}}_T, (\boldsymbol{\delta}^k_{\mathrm{D},TF}\underline{\boldsymbol{v}}_T)_{F \in \mathcal{F}_T}) := \underline{\boldsymbol{I}}^k_T \boldsymbol{r}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T - \underline{\boldsymbol{v}}_T$$

- The global Darcy bilinear form is assembled element-wise:

$$\boxed{\mathrm{a}_{\mathrm{D},h}(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_{\mathrm{D},T}(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)}$$

Pressure-velocity coupling

- The pressure-velocity coupling is realized by means of the bilinear

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := \sum_{T \in \mathcal{T}_h} \left((\boldsymbol{v}_T, \boldsymbol{\nabla} q_T)_T - \sum_{F \in \mathcal{F}_T} (\boldsymbol{v}_F, q_T \boldsymbol{n}_{TF})_F \right)$$

- Inf-sup stability: It holds, for all $q_h \in P_h^k$,

$$\boxed{\beta \|q_h\| \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k \setminus \{\underline{\boldsymbol{0}}\}} \frac{\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h)}{\|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{U},h}} \text{ with } \beta := (2\overline{\mu} + \overline{\nu})^{-\frac{1}{2}}}$$

Discrete problem and well-posedness

- Define the Stokes–Darcy global bilinear form

$$\mathrm{a}_h := \mathrm{a}_{\mathrm{S},h} + \mathrm{a}_{\mathrm{D},h}$$

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \sum_{T \in \mathcal{T}_h} (\boldsymbol{f}, \boldsymbol{r}_{\mathrm{D},T}^k \underline{\boldsymbol{v}}_T)_T \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= (g, q_h) && \forall q_h \in P_h^k \end{aligned}$$

Theorem (Well-posedness)

The discrete problem is well-posed with a priori bound:

$$\|\underline{\boldsymbol{u}}_h\|_{\boldsymbol{U},h} + \beta \|p_h\| \lesssim (2\underline{\mu})^{-\frac{1}{2}} \|\boldsymbol{f}\| + \beta^{-1} \|g\| \text{ with } \beta := (\overline{\mu} + \overline{\nu})^{-\frac{1}{2}} .$$

Convergence I

- We estimate the error $(\underline{\boldsymbol{e}}_h, \epsilon_h) \coloneqq (\underline{\boldsymbol{u}}_h - \underline{\hat{\boldsymbol{u}}}_h, p_h - \hat{p}_h)$ with

$$(\underline{\hat{\boldsymbol{u}}}_h, \hat{p}_h) \coloneqq (\underline{\boldsymbol{I}}_h^k \boldsymbol{u}, \pi_h^k p) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$$

- We have the following basic estimate [DP and Droniou, 2018]

$$\boxed{\|\underline{\boldsymbol{e}}_h\|_{\boldsymbol{U},h} + \beta\|\epsilon_h\| \lesssim \|\mathfrak{R}(\boldsymbol{u}, p)\|_{\boldsymbol{U}^*,h}}$$

with consistency error s.t., for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$,

$$\boxed{\langle \mathfrak{R}(\boldsymbol{u}, p), \underline{\boldsymbol{v}}_h\rangle \coloneqq (\boldsymbol{f}, \boldsymbol{r}_{\mathrm{D},h}^k \underline{\boldsymbol{v}}_h) - \mathrm{a}_h(\underline{\hat{\boldsymbol{u}}}_h, \underline{\boldsymbol{v}}_h) - \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \hat{p}_h)}$$

Convergence II

- For $T \in \mathcal{T}_h$, we identify the regime via the local friction coefficient

$$\boxed{C_{\mathrm{f},T} := \frac{\nu_T h_T^2}{2\mu_T} \text{ with } C_{\mathrm{f},T}^{-1} := +\infty \text{ if } \nu_T = 0}$$

- More precisely, we have
  - $C_{\mathrm{f},T} > 1$ if Darcy dominates (with pure Darcy if $C_{\mathrm{f},T} = +\infty$)
  - $C_{\mathrm{f},T} < 1$ if Stokes dominates (with pure Stokes if $C_{\mathrm{f},T}^{-1} = +\infty$)
  - ($C_{\mathrm{f},T} = 1$ for pure Brinkman)

Convergence III

Theorem (Estimate of the convergence rate)

Assuming $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h)^d$ *and* $p \in H^1(\Omega)$*, we have that*

$$\|\Re(\boldsymbol{u}, p)\|_{\boldsymbol{U}^*,h} \lesssim$$

$$h^{k+1} \left[\sum_{T \in \mathcal{T}_h} \left((2\mu_T) \min(1, C_{\mathrm{f},T}^{-1}) |\boldsymbol{u}|^2_{H^{k+2}(T)^d} + \nu_T \min(1, C_{\mathrm{f},T}) |\boldsymbol{u}|^2_{H^{k+1}(T)^d} \right) \right]^{\frac{1}{2}}.$$

This estimate extends to the pure Darcy case setting $C_{\mathrm{f},T} = +\infty$*.*

- Fully robust for $C_{\mathrm{f},T} \in [0, +\infty]$ thanks to the cut-off factors
- Equilibrated Stokes and Darcy contributions in $\mathcal{O}(h_T^{k+1})$
- **Bonus/1:** pressure-robust estimate for the velocity
- **Bonus/2:** $k = l = 0$ also works for Darcy ($C_{\mathrm{f},T} = +\infty$ for all $T \in \mathcal{T}_h$)

Static condensation

- Partition the discrete unknowns inside each $T \in \mathcal{T}_h$ as follows:
  - Velocity: element-based $\mathsf{U}_{\mathcal{T}_h}$ + face-based $\mathsf{U}_{\mathcal{F}_h^{\mathrm{i}}}$
  - Pressure: average value $\overline{\mathsf{P}}_{\mathcal{T}_h}$ + oscillations $\widetilde{\mathsf{P}}_{\mathcal{T}_h}$
- The linear system has the form

$$
\begin{bmatrix}
\mathbf{A}_{\mathcal{T}_h\mathcal{T}_h} & \widetilde{\mathbf{B}}_{\mathcal{T}_h} & \mathbf{A}_{\mathcal{T}_h\mathcal{F}_h^{\mathrm{i}}} & \overline{\mathsf{B}}_{\mathcal{T}_h} \\
\mathbf{A}_{\mathcal{F}_h^{\mathrm{i}}\mathcal{T}_h} & \widetilde{\mathbf{B}}_{\mathcal{F}_h^{\mathrm{i}}} & \mathbf{A}_{\mathcal{F}_h^{\mathrm{i}}\mathcal{F}_h^{\mathrm{i}}} & \overline{\mathsf{B}}_{\mathcal{F}_h^{\mathrm{i}}} \\
\widetilde{\mathbf{B}}_{\mathcal{T}_h}^{\mathrm{T}} & \mathbf{0} & \widetilde{\mathsf{B}}_{\mathcal{F}_h^{\mathrm{i}}}^{\mathrm{T}} & \mathbf{0} \\
\overline{\mathsf{B}}_{\mathcal{T}_h}^{\mathrm{T}} & 0 & \overline{\mathsf{B}}_{\mathcal{F}_h^{\mathrm{i}}}^{\mathrm{T}} & 0
\end{bmatrix}
\begin{bmatrix}
\mathsf{U}_{\mathcal{T}_h} \\ \widetilde{\mathsf{P}}_{\mathcal{T}_h} \\ \mathsf{U}_{\mathcal{F}_h^{\mathrm{i}}} \\ \overline{\mathsf{P}}_{\mathcal{T}_h}
\end{bmatrix}
=
\begin{bmatrix}
\mathsf{F}_{\mathcal{T}_h} \\ \widetilde{\mathsf{G}}_{\mathcal{T}_h} \\ \mathsf{F}_{\mathcal{F}_h} \\ \overline{\mathsf{G}}_{\mathcal{T}_h}
\end{bmatrix}
$$

- The matrix in red can be inexpensively inverte element-wise
- After **statically condensing** $\mathsf{U}_{\mathcal{T}_h}$ **and** $\widetilde{\mathsf{P}}_{\mathcal{T}_h}$, system of size

$$
\boxed{d\binom{k+d-1}{k}\operatorname{card}(\mathcal{F}_h^{\mathrm{i}}) + \operatorname{card}(\mathcal{T}_h)}
$$

Numerical examples I

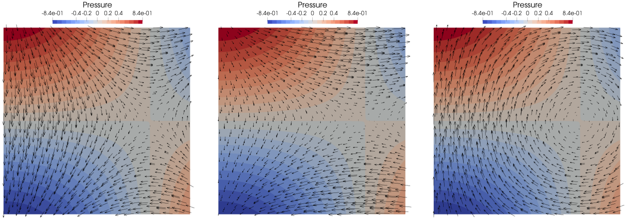

Figure: Velocity and pressure for Darcy ($C_{\mathrm{f},\Omega} = +\infty$), Brinkman ($C_{\mathrm{f},\Omega} = 1$), and Stokes ($C_{\mathrm{f},\Omega} = 0$)

We consider the exact solution parametrised by $C_{\mathrm{f},\Omega} \in [0, +\infty]$ s.t.

$$\boldsymbol{u}(\boldsymbol{x}) = \chi_{\mathrm{S}}\left(C_{\mathrm{f},\Omega}\right)\boldsymbol{u}_{\mathrm{S}}(\boldsymbol{x}) + (1 - \chi_{\mathrm{S}})\left(C_{\mathrm{f},\Omega}\right)\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{x}), \qquad p(\boldsymbol{x}) := \cos x_1 \sin x_2 - p_0,$$

where,

$$\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{x}) := \begin{cases} -\nu^{-1}\boldsymbol{\nabla} p(\boldsymbol{x}) & \text{if } \nu \neq 0, \\ \boldsymbol{0} & \text{otherwise,} \end{cases} \qquad \boldsymbol{u}_{\mathrm{S}}(\boldsymbol{x}) := -\operatorname{curl}(\sin x_1 \cos x_2)$$

Numerical examples II

Table: Convergence for Darcy

N_{dof}	N_{nz}	$\|\underline{\boldsymbol{e}}_h\|_{\boldsymbol{U},h}$	EOC	$\|\boldsymbol{e}_h\|$	EOC	$\|\epsilon_h\|$	EOC	τ_{ass}	τ_{sol}
				$k = 0$					
113	1072	1.69e-01	–	1.69e-01	–	1.39e-01	–	2.26e-03	9.68e-04
481	4944	8.84e-02	0.94	8.84e-02	0.94	4.27e-02	1.70	1.19e-02	5.34e-03
1985	21136	4.47e-02	0.98	4.47e-02	0.98	1.18e-02	1.86	3.34e-02	5.83e-02
8065	87312	2.22e-02	1.01	2.22e-02	1.01	3.69e-03	1.67	1.12e-01	1.02e+00
32513	354832	1.09e-02	1.03	1.09e-02	1.03	1.45e-03	1.35	3.94e-01	3.39e+01
				$k = 1$					
193	3456	1.33e-02	–	3.89e-03	–	5.15e-03	–	4.24e-03	1.71e-03
833	16192	2.65e-03	2.32	7.73e-04	2.33	1.01e-03	2.36	1.98e-02	1.91e-02
3457	69696	6.55e-04	2.02	1.90e-04	2.03	2.27e-04	2.15	6.16e-02	1.35e-01
14081	288832	1.66e-04	1.98	4.80e-05	1.98	5.53e-05	2.03	2.05e-01	1.94e+00
56833	1175616	4.32e-05	1.94	1.25e-05	1.94	1.37e-05	2.01	7.70e-01	6.49e+01
				$k = 2$					
273	7216	4.84e-03	–	1.25e-03	–	2.48e-04	–	7.61e-03	2.57e-03
1185	34000	7.55e-04	2.68	1.94e-04	2.68	2.94e-05	3.08	3.64e-02	4.46e-02
4929	146704	1.00e-04	2.91	2.59e-05	2.90	3.76e-06	2.97	1.23e-01	2.39e-01
20097	608656	1.29e-05	2.95	3.36e-06	2.95	4.77e-07	2.98	4.02e-01	3.84e+00
81153	2478736	1.64e-06	2.98	4.25e-07	2.98	5.94e-08	3.00	1.55e+00	8.75e+01

Numerical examples III

Table: Convergence for Brinkman

N_{dof}	N_{nz}	$\|\underline{\boldsymbol{e}}_h\|_{\boldsymbol{U},h}$	EOC	$\|\boldsymbol{e}_h\|$	EOC	$\|\epsilon_h\|$	EOC	τ_{ass}	τ_{sol}
				$k = 1$					
193	3456	6.48e-02	–	3.51e-03	–	3.40e-02	–	4.86e-03	1.87e-03
833	16192	2.78e-02	1.22	7.40e-04	2.24	9.34e-03	1.86	1.65e-02	2.05e-02
3457	69696	8.93e-03	1.64	1.18e-04	2.65	2.60e-03	1.84	6.32e-02	1.19e-01
14081	288832	2.43e-03	1.88	1.62e-05	2.87	6.84e-04	1.93	2.20e-01	1.69e+00
56833	1175616	6.30e-04	1.95	2.10e-06	2.95	1.75e-04	1.97	8.13e-01	4.38e+01
				$k = 2$					
273	7216	3.72e-03	–	1.21e-04	–	1.74e-03	–	8.64e-03	2.76e-03
1185	34000	7.56e-04	2.30	1.24e-05	3.28	1.98e-04	3.13	3.56e-02	3.12e-02
4929	146704	1.13e-04	2.74	9.35e-07	3.73	2.29e-05	3.12	1.28e-01	1.87e-01
20097	608656	1.52e-05	2.89	6.30e-08	3.89	2.70e-06	3.08	4.23e-01	2.97e+00
81153	2478736	1.96e-06	2.95	4.08e-09	3.95	3.27e-07	3.04	1.71e+00	5.92e+01
				$k = 3$					
353	12352	2.44e-04	–	6.48e-06	–	1.41e-04	–	1.74e-02	3.93e-03
1537	58368	1.99e-05	3.62	2.68e-07	4.60	9.32e-06	3.92	7.41e-02	4.50e-02
6401	252160	1.27e-06	3.97	8.50e-09	4.98	5.65e-07	4.04	2.53e-01	4.28e-01
26113	1046784	8.26e-08	3.94	2.79e-10	4.93	3.58e-08	3.98	9.11e-01	5.58e+00
105473	4264192	5.19e-09	3.99	8.78e-12	4.99	2.23e-09	4.00	3.67e+00	8.72e+01

Numerical examples IV

Table: Convergence for Stokes

N_{dof}	N_{nz}	$\\|\underline{\boldsymbol{e}}_h\\|_{\boldsymbol{U},h}$	EOC	$\\|\boldsymbol{e}_h\\|$	EOC	$\\|\epsilon_h\\|$	EOC	τ_{ass}	τ_{sol}
				$k = 1$					
193	3456	1.10e-02	–	6.07e-04	–	1.82e-02	–	6.74e-03	2.36e-03
833	16192	3.79e-03	1.54	1.09e-04	2.48	5.06e-03	1.85	1.61e-02	2.31e-02
3457	69696	1.04e-03	1.86	1.52e-05	2.84	1.32e-03	1.94	7.64e-02	1.33e-01
14081	288832	2.71e-04	1.94	1.99e-06	2.93	3.37e-04	1.96	2.32e-01	1.68e+00
56833	1175616	6.98e-05	1.96	2.56e-07	2.96	8.53e-05	1.98	8.35e-01	4.41e+01
				$k = 2$					
273	7216	1.38e-03	–	4.97e-05	–	1.70e-03	–	9.99e-03	2.82e-03
1185	34000	1.95e-04	2.83	3.47e-06	3.84	2.39e-04	2.83	4.15e-02	3.44e-02
4929	146704	2.74e-05	2.83	2.39e-07	3.86	3.06e-05	2.96	2.38e-01	2.09e-01
20097	608656	3.58e-06	2.94	1.55e-08	3.94	3.90e-06	2.97	4.52e-01	3.11e+00
81153	2478736	4.50e-07	2.99	9.77e-10	3.99	4.90e-07	2.99	1.74e+00	6.17e+01
				$k = 3$					
353	12352	1.17e-04	–	3.38e-06	–	1.51e-04	–	1.78e-02	4.03e-03
1537	58368	8.48e-06	3.79	1.26e-07	4.74	1.07e-05	3.83	7.66e-02	4.63e-02
6401	252160	5.43e-07	3.96	4.01e-09	4.98	6.70e-07	3.99	2.58e-01	4.51e-01
26113	1046784	3.45e-08	3.98	1.28e-10	4.97	4.24e-08	3.98	9.33e-01	5.87e+00
105473	4264192	2.18e-09	3.99	4.04e-12	4.99	2.66e-09	3.99	3.63e+00	9.27e+01

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