An introduction to the convergence analysis of discretisation methods for PDEs with application to Hybrid High-Order methods

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A model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denote an open bounded connected polytopal set
- Let $f : \overline{\Omega} \to \mathbb{R}$ denote a given source term
- We consider the Poisson problem: Find $u : \overline{\Omega} \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

where we recall that the Laplace operator is defined as

$$\Delta u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$$

Advantages of the weak formulation

- Let, for the moment being, $d = 1$ and $\Omega = (0, 1)$
- The Poisson problem reads in this case: Find $u : [0, 1] \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= f && \text{in } (0,1), \\
u(0) &= u(1) = 0
\end{aligned}
$$

- This problem is meaningful if $f \in C^0([0, 1])$ and $u \in C^2([0, 1])$
- This is however not representative of real-life problems, where the source term can be discontinuous!
- The weak formulation covers this (and other) important case(s)

Weak derivatives I

- For any function $\phi \in C^\infty(\Omega)$, we define its support by

$$\operatorname{supp}(\phi) := \overline{\{x \in \Omega \,:\, \phi(x) \neq 0\}}$$

- Denote by $C_0^\infty(\Omega)$ the set of functions with compact support in Ω

$$C_0^\infty(\Omega) := \{\phi \in C^\infty(\Omega) \,:\, \operatorname{supp}(\phi) \text{ is a compact subset of } \Omega\},$$

  i.e., functions in $C_0^\infty(\Omega)$ vanish near $\partial\Omega$

- We define the set of locally Lebesgue integrable functions

$$L^1_{\text{loc}}(\Omega) := \left\{ f \,:\, \int_K |f(x)| \mathrm{d}x < +\infty \text{ for all compact } K \subset \Omega \right\}$$

Weak derivatives II

Definition (Weak first partial derivative and weak gradient)

We say that $v \in L^1_{\text{loc}}(\Omega)$ has weak partial derivative w.r. to the ith variable if there exists $w \in L^1_{\text{loc}}(\Omega)$ s.t.

$$\int_\Omega w(x)\phi(x)\mathrm{d}x = -\int_\Omega v(x)\frac{\partial \phi(x)}{\partial x_i}\mathrm{d}x \qquad \forall \phi \in C_0^\infty(\Omega)$$

and we set

$$\partial_i v := w.$$

If v function has weak partial derivatives with respect to the ith variable for any $1 \leq i \leq d$, we define its weak gradient

$$\nabla v := \begin{pmatrix} \partial_i v \\ \vdots \\ \partial_d v \end{pmatrix}$$

Hilbert spaces I

Definition (Inner product space)

A inner product space is a vector space V over $\mathbb{R}$ together with an inner product $(\cdot,\cdot)_V$, i.e., a map $(\cdot,\cdot)_V : V \times V \longrightarrow \mathbb{R}$ s.t., for all $(u, v, z) \in U^3$ and all $\alpha \in \mathbb{R}$, the following properties hold:

$$
\begin{aligned}
(u, v)_V &= (v, u)_V, && \text{(Symmetry)}\\
(\alpha u, v)_V &= \alpha(u, v)_V \text{ and } (u + v, z)_V = (u, z)_V + (v, z)_V, && \text{(Linearity)}\\
(v, v)_V &\geq 0 \text{ and } (v, v)_V = 0 \text{ iff } v = 0. && \text{(Positivity)}
\end{aligned}
$$

We denote by $\|\cdot\|_V$ the norm induced by the inner product on V.

Lemma (Cauchy–Schwarz inequality)

Let $(V, (\cdot,\cdot)_V)$ be an inner-product space. Then, for all $u, v \in V$,

$$|(u, v)_V| \leq \|u\|_V \ \|v\|_V .$$

A similar inquality is valid for any positive semi-definite bilinear form on $V \times V$.

Hilbert spaces II

Definition (Hilbert space)

A Hilbert space is an inner product space $(V, (\cdot,\cdot)_V)$ that is complete with respect to the distance function defined by the norm, i.e., every Cauchy sequence converges in V.

We recall that a Cauchy sequence in this context is a sequence $(\phi_n)_{n \in N} \in V^{\mathbb{N}}$ s.t., for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t., for all $n, m \geq N$, $\|\phi_m - \phi_n\|_V < \epsilon$.

The space of finite energy functions for the Poisson problem I

- Let $\|\cdot\|_{L^2(\Omega)}$ map a given function $v : \Omega \to \mathbb{R}$ on

$$\|v\|_{L^2(\Omega)} \coloneqq \left(\int_\Omega |v(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

- We define the Lebesgue space of square-integrable functions

$$L^2(\Omega) \coloneqq \left\{v : \Omega \to \mathbb{R} \ : \ \|v\|_{L^2(\Omega)} < +\infty\right\}$$

- The space of finite energy functions for the Poisson problem is

$$H^1(\Omega) \coloneqq \left\{v \in L^2(\Omega) \ : \ \partial_i v \in L^2(\Omega) \quad \forall 1 \le i \le d\right\}$$

The space of finite energy functions for the Poisson problem II

- We equip $H^1(\Omega)$ with the following inner product:

$$(u, v)_{H^1(\Omega)} := \int_\Omega u(x)v(x)\mathrm{d}x + \int_\Omega \nabla u(x) \cdot \nabla v(x)\mathrm{d}x$$

- The corresponding norm is

$$\|v\|_{H^1(\Omega)} := \left(\|v\|_{L^2(\Omega)}^2 + |v|_{H^1(\Omega)}^2\right)^{\frac{1}{2}} \quad \text{with} \quad |v|_{H^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)^d}$$

- It can be proved that $(H^1(\Omega), (\cdot, \cdot)_{H^1(\Omega)})$ is a Hilbert space

Boundary conditions and Poincaré inequality

- Finite energy functions that vanish on $\partial\Omega$ are collected in the space

$$H_0^1(\Omega) := \left\{ v \in H^1(\Omega) \; : \; v_{|\partial\Omega} = 0 \right\}$$

- A crucial result is the following Poincare inequality: There exists C_Ω only depending on Ω s.t.

$$\|v\|_{L^2(\Omega)} \leq C_\Omega |v|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

- As a result, $|\cdot|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$

Weak formulation

- Let $f \in L^2(\Omega)$, which includes possibly discontinuous source terms
- Set $U := H_0^1(\Omega)$ and let $a : U \times U \to \mathbb{R}$ and $\ell : U \to \mathbb{R}$ be s.t.

$$a(u,v) := \int_\Omega u(x)v(x)\mathrm{d}x, \qquad \ell(v) := \int_\Omega f(x)\mathrm{d}x$$

- The weak formulation of our model problem reads:

$$\text{Find } u \in U \text{ s.t.} \quad a(u,v) = \ell(v) \quad \forall v \in U$$

- It can be proved that u minimises the energy

$$\Phi(v) := \frac{1}{2}a(v,v) - \ell(v)$$

Well-posedness I

Lemma (Lax–Milgram)

Given a Hilbert space $(V, (\cdot,\cdot)_V)$, assume that there exist strictly positive real numbers α, γ, and L s.t.

$$
\begin{aligned}
\alpha\|v\|_V^2 &\le a(v,v) && \forall v \in V, && \textit{(Coercivity)}\\
|a(u,v)| &\le \gamma\|u\|_V\|v\|_V && \forall (u,v) \in V^2, && \textit{(Boundedness of } a\textit{)}\\
|\ell(v)| &\le L\|v\|_V && \forall v \in V. && \textit{(Boundedness of } \ell\textit{)}
\end{aligned}
$$

Then, the problem:

$$
\textit{Find } u \in V \textit{ s.t.} \quad a(u,v) = \ell(v) \quad \forall v \in V
$$

admits a unique solution which satisfies the a priori estimate

$$
\|v\|_V \le \frac{L}{\alpha}.
$$

Well-posedness II

Theorem (Well-posedness of the Poisson problem)

The Poisson problem is well-posed, and it holds

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{1 + C_\Omega^2} \|f\|_{L^2(\Omega)}.$$

Well-posedness III

- Using Poincare's inequality, we have for all $v \in U$,

$$\frac{1}{1+C_\Omega^2}\|u\|_{H^1(\Omega)}^2 = \frac{1}{1+C_\Omega^2}\left(\|v\|_{L^2(\Omega)}^2 + |v|_{H^1(\Omega)}^2\right) \le \|\nabla v\|_{L^2(\Omega)^d}^2 = a(v,v),$$

  that is, a is coercive with $\alpha = 1/(1+C_\Omega^2)$
- Moreover, for all $(u,v) \in U^2$, using the Cauchy–Schwarz inequality,

$$|a(u,v)| \le \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

  i.e., a is bounded with $\gamma = 1$
- Finally, using again Poincaré's inequality, for all $v \in U$

$$|\ell(v)| = \left|\int_\Omega f(x)v(x)\mathrm{d}x\right| \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

  which shows that ℓ is bounded with $L = \|f\|_{L^2(\Omega)}$

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Setting

Definition (Continuous problem)

Let a Hilbert space H, a continuous bilinear form $a : H \times H \longrightarrow \mathbb{R}$, and a continuous linear form $\ell : H \longrightarrow \mathbb{R}$ be given. The problem we aim at approximating is

$$\text{Find } u \in H \text{ s.t. } \quad a(u, v) = \ell(v) \quad \forall v \in H. \tag{Π}$$

Definition (Discrete problem)

Let a vector space X_h with norm $\|\cdot\|_{X_h}$, a bilinear form $a_h : X_h \times X_h \longrightarrow \mathbb{R}$, and a linear form $\ell_h : X_h \longrightarrow \mathbb{R}$ be given. The approximation of problem (Π) is

$$\text{Find } u_h \in X_h \text{ s.t. } \quad a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in X_h \tag{Π_h}$$

with h discretisation parameter s.t. convergence is expected when $h \to 0$.

Stability

Definition (Coercivity)

The bilinear form a_h is coercive for $\|\cdot\|_{X_h}$ if

$$\exists \gamma > 0 \text{ s.t. } \gamma \|v_h\|_{X_h}^2 \leq a_h(v_h, v_h) \quad \forall v_h \in X_h.$$

A more general notion of stability is the following:

Definition (Inf–sup stability)

The bilinear form a_h is inf–sup stable for $\|\cdot\|_{X_h}$ if

$$\exists \gamma > 0 \text{ s.t. } \gamma \|u_h\|_{X_h} \leq \sup_{v_h \in X_h \setminus \{0\}} \frac{a_h(u_h, v_h)}{\|v_h\|_{X_h}} \quad \forall u_h \in X_h.$$

Remark

For optimal error estimates, one usually needs γ to be independent of h.

A priori bound on the discrete solution

Proposition (A priori bound on the discrete solution)

If a_h is inf–sup stable, $m_h : X_h \to \mathbb{R}$ is linear, and w_h satisfies

$$a_h(w_h, v_h) = m_h(v_h) \quad \forall v_h \in X_h,$$

then, setting $\|m_h\|_{X_h^\star} := \sup_{v_h \in X_h \setminus \{0\}} \frac{|m_h(v_h)|}{\|v_h\|_{X_h}}$,

$$\|w_h\|_{X_h} \leq \gamma^{-1} \|m_h\|_{X_h^\star}.$$

Proof.

Take $v_h \in X_h \setminus \{0\}$ and write, by definition of $\|\cdot\|_{X_h^\star}$,

$$\frac{a_h(w_h, v_h)}{\|v_h\|_{X_h}} = \frac{m_h(v_h)}{\|v_h\|_{X_h}} \leq \|m_h\|_{X_h^\star}.$$

The proof is completed by taking the supremum over such v_h. □

Consistency error and consistency

Definition (Consistency error and consistency)

Let u solve (Π) and take $I_h u \in X_h$. The variational consistency error is the linear form $\mathcal{E}_h(u;\cdot) : X_h \to \mathbb{R}$ defined by

$$\mathcal{E}_h(u;\cdot) = \ell_h(\cdot) - a_h(I_h u, \cdot).$$

Let now a family $(X_h, a_h, \ell_h)_{h \to 0}$ of spaces and forms be given, and consider the corresponding discrete problems (Π_h). Consistency holds if

$$\|\mathcal{E}_h(u;\cdot)\|_{X_h^\star} \to 0 \text{ as } h \to 0.$$

Remark (Choice of $I_h u$)

No particular property is required here on $I_h u$; it could actually be any element of X_h. However, for the estimates that follow to be meaningful, it is expected that $I_h u$ is computed from u, so that information on $I_h u$ encodes meaningful information on u itself.

Abstract energy error estimate I

Theorem (Abstract energy error estimate and convergence)

Assume a_h inf–sup stable. Let u be a solution to (Π) and $I_h u \in X_h$. If u_h is a solution to (Π_h) then

$$\|u_h - I_h u\|_{X_h} \leq \gamma^{-1} \|\mathcal{E}_h(u; \cdot)\|_{X_h^\star}.$$

As a consequence, letting a family $(X_h, a_h, \ell_h)_{h\to 0}$ of spaces and forms be given, if consistency holds, then we have convergence in the following sense:

$$\|u_h - I_h u\|_{X_h} \to 0 \text{ as } h \to 0.$$

Abstract energy error estimate II

Proof.

For any $v_h \in X_h$, the scheme (Π_h) yields

$$a_h(u_h - I_h u, v_h) = a_h(u_h, v_h) - a_h(I_h u, v_h) = \ell_h(v_h) - a_h(I_h u, v_h).$$

Recalling the definition of the consistency error, the error $u_h - I_h u$ can then be characterised as the solution to the following error equation:

$$a_h(u_h - I_h u, v_h) = \mathcal{E}_h(u; v_h) \qquad \forall v_h \in X_h. \qquad (\Pi_{\text{err},h})$$

The proof is completed by writing the a priori bound with $m_h = \mathcal{E}_h(u; \cdot)$ and $w_h = u_h - I_h u$. □

Quasi-optimality of the error estimate

Remark

Let

$$\|a_h\|_{X_h\times X_h} := \sup_{w_h\in X_h\setminus\{0\}, v_h\in Y_h\setminus\{0\}} \frac{|a_h(w_h,v_h)|}{\|w_h\|_{X_h}\|v_h\|_{X_h}}$$

be the standard norm of the bilinear form a_h. The error equation $(\Pi_{\mathrm{err},h})$ shows that

$$\|\mathcal{E}_h(u;\cdot)\|_{X_h^\star} \leq \|a_h\|_{X_h\times X_h}\|u_h - I_h u\|_{X_h}.$$

Hence, if $\|a_h\|_{X_h\times X_h}$ and γ remain bounded with respect to h as $h\to 0$, which is always the case in practice, the error estimate is quasi-optimal in the sense that, for some C not depending on h, it holds that

$$C^{-1}\|\mathcal{E}_h(u;\cdot)\|_{X_h^\star} \leq \|u_h - I_h u\|_{X_h} \leq C\|\mathcal{E}_h(u;\cdot)\|_{X_h^\star}.$$

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Features

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Robustness with respect to the variations of the physical coefficients
- Reduced computational cost after hybridization

$$N_{\mathrm{dof},h} = \mathrm{card}(\mathcal{F}_h^{\mathrm{i}})\binom{k+d-1}{d-1}$$

Polyhedral meshes

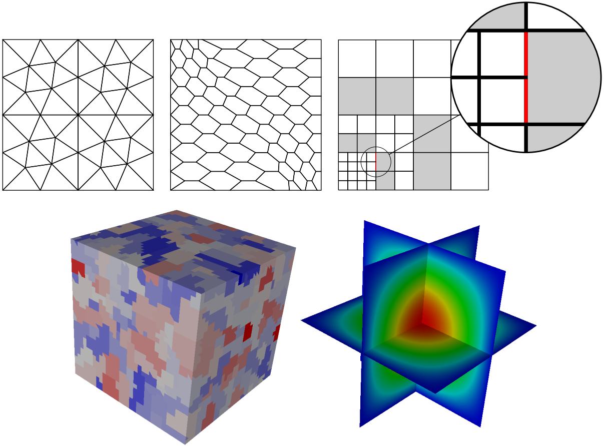


Figure: Admissible meshes in 2d and 3d, and HHO solution on the agglomerated 3d mesh

Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, as before
- For $X \subset \Omega$, we denote by $(\cdot, \cdot)_X$ the standard inner product of $L^2(X)$ and set $\|v\|_X := \sqrt{(v, v)_X}$. When $X = \Omega$, the subscript is omitted
- We come back to the Poisson problem: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

- Hereafter, $a \lesssim b$ means $a \leq Cb$ with C independent of h. $a \simeq b$ means $a \lesssim b \lesssim a$

Sobolev spaces

- For all $p \in [1, +\infty]$ we set, for all $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^d$,

$$\|\boldsymbol{x}\|_p := \begin{cases} \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < +\infty, \\ \max_{1 \le i \le d} |x_i| & \text{if } p = +\infty. \end{cases}$$

- Let $X \subset \mathbb{R}^d$. For all $s \in \mathbb{N}$, we define the Sobolev space

$$H^s(X) := \left\{ v \in L^2(X) \ : \ \forall \boldsymbol{\alpha} \in \boldsymbol{A}_d^s, \ \partial^{\boldsymbol{\alpha}} v \in L^2(X) \right\}$$

with $\boldsymbol{A}_d^s := \left\{ \boldsymbol{\alpha} \in \mathbb{N}^d \ : \ \|\boldsymbol{\alpha}\|_1 \le s \right\}$. By definition,

$$H^0(X) = L^2(X)$$

- The Sobolev norm $\|\cdot\|_{W^{s,p}(X)}$ and seminorm $|\cdot|_{W^{s,p}(X)}$ are

$$\|v\|_{H^s(X)} := \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}_d^s} \|\partial^{\boldsymbol{\alpha}} v\|_X, \quad |v|_{H^s(X)} := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n, \|\boldsymbol{\alpha}\|_1 = s} \|\partial^{\boldsymbol{\alpha}} v\|_X$$

Projectors on local polynomial spaces I

- At the core of HHO are projectors on local polynomial spaces
- With $X = T$ or $X = F$, the L^2-projector $\pi_X^{0,l} : L^1(T) \longrightarrow \mathbb{P}^l(X)$ is s.t.

$$(\pi_X^{0,l} v - v, w)_X = 0 \text{ for all } w \in \mathbb{P}^l(X)$$

- The elliptic projector $\pi_T^{1,l} : W^{1,1}(T) \longrightarrow \mathbb{P}^l(T)$ is s.t.

$$(\boldsymbol{\nabla}(\pi_T^{1,l} v - v), \boldsymbol{\nabla} w)_T = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } (\pi_T^{1,l} v - v, 1)_T = 0$$

Projectors on local polynomial spaces II

Theorem (Optimal approximation properties of projectors)

For $\xi \in \{0,1\}$ and $s \in \{\xi, \ldots, l+1\}$, it holds for all $T \in \mathcal{T}_h$ and $v \in H^s(T)$,

$$|v - \pi_T^{\xi,l} v|_{H^m(T)} \lesssim h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s\},$$

and, if $s \geq 1$,

$$|v - \pi_T^{\xi,l} v|_{H^m(\mathcal{F}_T)} \lesssim h_T^{s-m-\frac{1}{2}} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s-1\},$$

where $H^m(\mathcal{F}_T) := \left\{ v \in L^2(\partial T) \, : \, v_{|F} \in H^m(F) \quad \forall F \in \mathcal{F}_T \right\}$ is the broken Sobolev space on the boundary of T.

Proof.

See [Di Pietro and Droniou, 2017a, Di Pietro and Droniou, 2017b]. □

Computing $\pi_T^{1,k+1}$ from L^2-projections of degree k

- The following integration by parts formula is valid for all $w \in C^\infty(\overline{T})$:

$$\boxed{(\nabla v, \nabla w)_T = -(v, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (v, \nabla w \cdot \boldsymbol{n}_{TF})_F}$$

- Specializing it to $w \in \mathbb{P}^{k+1}(T)$, we can write

$$(\nabla \pi_T^{1,k+1} v, \nabla w)_T = -(\pi_T^{0,k} v, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (\pi_F^{0,k} v_{|F}, \nabla w \cdot \boldsymbol{n}_{TF})_F$$

- Moreover, it can be easily seen that

$$(\pi_T^{1,k+1} v - v, 1)_T = (\pi_T^{1,k+1} v - \pi_T^{0,k} v, 1)_T = 0$$

- **Hence, $\pi_T^{1,k+1} v$ can be computed from $\pi_T^{0,k} v$ and $(\pi_F^{0,k} v_{|F})_{F \in \mathcal{F}_T}$!**

Discrete unknowns

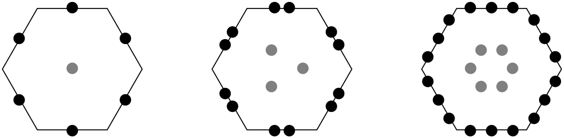


Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- Let a polynomial degree $k \geq 0$ be fixed
- For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\underline{U}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \, : \, v_T \in \mathbb{P}^k(T) \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v := (\pi_T^{0,k} v, (\pi_F^{0,k} v_{|F})_{F \in \mathcal{F}_T})$$

Local potential reconstruction

- Let $T \in \mathcal{T}_h$. We define the local potential reconstruction operator

$$r_T^{k+1} : \underline{U}_T^k \longrightarrow \mathbb{P}^{k+1}(T)$$

s.t. for all $\underline{v}_T \in \underline{U}_T^k$, $(r_T^{k+1}\underline{v}_T - v_T, 1)_T = 0$ and

$$\boxed{(\nabla r_T^{k+1}\underline{v}_T, \nabla w)_T = -(v_T, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla w \cdot \boldsymbol{n}_{TF})_F \quad \forall w \in \mathbb{P}^{k+1}(T)}$$

- By construction, we have

$$\boxed{r_T^{k+1} \circ \underline{I}_T^k = \pi_T^{1,k+1}}$$

- $r_T^{k+1} \circ \underline{I}_T^k$ has therefore optimal approximation properties in $\mathbb{P}^{k+1}(T)$

Stabilization I

- We would be tempted to approximate

$$a_{|T}(u,v) \approx (\nabla r_T^{k+1} \underline{u}_T, \nabla r_T^{k+1} \underline{v}_T)_T$$

- This choice, however, is not stable in general. We consider instead

$$\boxed{\mathrm{a}_T(\underline{u}_T, \underline{v}_T) := (\nabla r_T^{k+1} \underline{u}_T, \nabla r_T^{k+1} \underline{v}_T)_T + \mathrm{s}_T(\underline{u}_T, \underline{v}_T)}$$

- The role of s_T is to ensure $\|\cdot\|_{1,T}$-coercivity with

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \|v_F - v_T\|_F^2 \quad \forall \underline{v}_T \in \underline{U}_T^k$$

Stabilization II

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ satisfies the following properties:

(S1) Symmetry and positivity. s_T is symmetric and positive semidefinite.

(S2) Stability. It holds, with hidden constant independent of h and T,

$$\mathrm{a}_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \simeq \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

(S3) Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{v}_T \in \underline{U}_T^k$,

$$\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0.$$

Stabilization III

Proposition (Consistency of s_T)

Let $T \in \mathcal{T}_h$ and let s_T denote a stabilisation bilinear form satisfying assumptions (S1)–(S3). Let $r \in \{0, \ldots, k\}$. Then, there is a real number $C > 0$ independent of both h and T s.t., for all $v \in H^{r+2}(T)$,

$$\mathrm{s}_T(\underline{I}_T^k v, \underline{I}_T^k v)^{\frac{1}{2}} \leq C h_T^{r+1} |v|_{H^{r+2}(T)}.$$

Stabilization IV

- The following stable choice violates polynomial consistency:

$$\mathrm{s}_T^{\mathrm{hdg}}(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (u_F - u_T, v_F - v_T)_F$$

- To circumvent this problem, we penalize the high-order differences

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) := \underline{I}_T^k r_T^{k+1} \underline{v}_T - \underline{v}_T$$

- The classical HHO stabilization bilinear form reads

$$\boxed{\mathrm{s}_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} ((\delta_T^k - \delta_{TF}^k) \underline{u}_T, (\delta_T^k - \delta_{TF}^k) \underline{v}_T)_F}$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{U}_h^k := \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) : \\ v_T \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h \text{ and } v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_h \Big\}$$

  and its subspace with strongly enforced boundary conditions

$$\underline{U}_{h,0}^k := \{ \underline{v}_h \in \underline{U}_h^k : v_F \equiv 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$\boxed{\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{u}_T, \underline{v}_T) = \sum_{T\in\mathcal{T}_h} (f, v_T)_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k}$$

- Well-posedness follows from coercivity and discrete Poincaré

Properties of a_h I

Lemma (Properties of a_h)

The bilinear form a_h *enjoys the following properties:*

(i) *Stability and boundedness. For all* $\underline{v}_h \in \underline{U}_{h,0}^k$ *it holds*

$$\|\underline{v}_h\|_{1,h} \simeq \|\underline{v}_h\|_{\mathrm{a},h} \text{ with } \|\underline{v}_h\|_{\mathrm{a},h} := \mathrm{a}_h(\underline{v}_h, \underline{v}_h)^{\frac{1}{2}}.$$

(ii) *Consistency. For all* $r \in \{0, \ldots, k\}$ *and* $w \in H_0^1(\Omega) \cap H^{r+2}(\Omega)$ *s.t.* $\Delta w \in L^2(\Omega)$,

$$\sup_{\underline{v}_h \in \underline{U}_{h,0}^k, \|\underline{v}_h\|_{\mathrm{a},h}=1} |\mathcal{E}_h(w; \underline{v}_h)| \lesssim h^{r+1} |w|_{H^{r+2}(\Omega)},$$

where the hidden constant is independent of w *and* h*, and the linear form* $\mathcal{E}_h(w; \cdot) : \underline{U}_{h,0}^k \to \mathbb{R}$ *representing the conformity error is s.t.*

$$\mathcal{E}_h(w; \underline{v}_h) := -(\Delta w, v_h) - \mathrm{a}_h(\underline{I}_h^k w, \underline{v}_h).$$

Properties of a_h II

- Point (i) is an immediate consequence of the assumptions on s_T
- Let $\underline{v}_h \in \underline{U}_{h,0}^k$ be s.t. $\|\underline{v}_h\|_{\mathrm{a},h} = 1$. For the sake of brevity, we let

$$\check{w}_T := r_T^{k+1} \underline{I}_T^k w_{|T} = \pi_T^{1,k+1} w_{|T} \quad \forall T \in \mathcal{T}_h$$

- Integrating by parts element by element, we infer that

$$\begin{aligned}
-(\Delta w, v_h) &= \sum_{T \in \mathcal{T}_h} \left((\nabla w, \nabla v_T)_T - \sum_{F \in \mathcal{F}_T} (\nabla w \cdot \boldsymbol{n}_{TF}, v_T)_F \right) \\
&= \sum_{T \in \mathcal{T}_h} \left((\nabla w, \nabla v_T)_T + \sum_{F \in \mathcal{F}_T} (\nabla w \cdot \boldsymbol{n}_{TF}, v_F - v_T)_F \right)
\end{aligned}$$

- To insert v_F into the second term, we have used the fact that $v_F = 0$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$ while, for all $F \in \mathcal{F}_h^{\mathrm{i}}$ s.t. $F \subset \partial T_1 \cap \partial T_2$, $T_1 \neq T_2$,

$$(\nabla w)_{|T_1} \cdot \boldsymbol{n}_{T_1 F} + (\nabla w)_{|T_2} \cdot \boldsymbol{n}_{T_2 F} = 0$$

Properties of a_h III

- Expanding a_T then $r_T^{k+1}\underline{v}_T$ according to the respective definitions, we get

$$\mathrm{a}_h(\underline{I}_h^k w, \underline{v}_h) = \sum_{T\in\mathcal{T}_h}\left((\boldsymbol{\nabla}\check{w}_T, \boldsymbol{\nabla}v_T)_T + \sum_{F\in\mathcal{F}_T}(\boldsymbol{\nabla}\check{w}_T\cdot\boldsymbol{n}_{TF}, v_F - v_T)_F\right) + \mathrm{s}_h(\underline{I}_h^k w, \underline{v}_h)$$

- Combining the above relations, we get

$$\begin{aligned}
&|\mathcal{E}_h(w;\underline{v}_h)| \\
&\quad= \left|\sum_{T\in\mathcal{T}_h}\left(\sum_{F\in\mathcal{F}_T}(\boldsymbol{\nabla}(w-\check{w}_T)\cdot\boldsymbol{n}_{TF}, v_F - v_T)_F\right) + \mathrm{s}_h(\underline{I}_h^k w, \underline{v}_h)\right| \\
&\quad\le \left|\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{\frac12}\|\boldsymbol{\nabla}(w-\check{w}_T)\|_F\, h_F^{-\frac12}\|v_F - v_T\|_F\right| + |\mathrm{s}_h(\underline{I}_h^k w, \underline{v}_h)|
\end{aligned}$$

Properties of a_h IV

- Repeated applications of the Cauchy–Schwarz inequality give

$$|\mathcal{E}_h(w;\underline{v}_h)| \le \left(\sum_{T\in\mathcal{T}_h} h_T \|\boldsymbol{\nabla}(w-\check{w}_T)\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} |\underline{v}_T|_{1,\partial T}^2\right)^{\frac{1}{2}} + \mathrm{s}_h(\underline{I}_h^k w, \underline{I}_h^k w)^{\frac{1}{2}} \mathrm{s}_h(\underline{v}_h, \underline{v}_h)^{\frac{1}{2}}$$

- Using the approximation properties of $\pi_T^{1,k+1}$ and of s_T, we infer

$$|\mathcal{E}_h(w;\underline{v}_h)| \lesssim h^{r+1}|w|_{H^{r+2}(\Omega)} \left[\left(\sum_{T\in\mathcal{T}_h} |\underline{v}_T|_{1,\partial T}^2\right)^{\frac{1}{2}} + |\underline{v}_h|_{\mathrm{s},h}\right]$$

- Recalling that $\|\underline{v}_h\|_{\mathrm{a},h} = 1$, the definition of a_h and the coercivity property in point (i), the terms involving $\underline{v}_T$ and $\underline{v}_h$ above are bounded by a constant independent of h and point (ii) follows $\square$

Convergence I

Theorem (Energy error estimate)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence. Let a polynomial degree $k \geq 0$ be fixed. Let $u \in H_0^1(\Omega)$ denote the exact solution, for which we assume the additional regularity $u \in H^{r+2}(\Omega)$ for some $r \in \{0, \ldots, k\}$. For all $h \in \mathcal{H}$, let $\underline{u}_h \in \underline{U}_{h,0}^k$ denote the discrete solution with stabilisation bilinear forms s_T, $T \in \mathcal{T}_h$, satisfying assumptions (S1)–(S3). Then,

$$\|\underline{u}_h - \underline{I}_h^k u\|_{\mathrm{a},h} \lesssim h^{r+1} |u|_{H^{r+2}(\Omega)}$$

where the hidden constant is independent of h and u.

Proof.

We invoke the abstract result with $H = H_0^1(\Omega)$, $a(u, v) = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)$, $\ell(v) = (f, v)$, $X_h = \underline{U}_{h,0}^k$ endowed with the norm $\|\cdot\|_{\mathrm{a},h}$, $a_h = \mathrm{a}_h$, $\ell_h(\underline{v}_h) = (f, v_h)$ and $I_h u = \underline{I}_h^k u$. We notice that a_h is obviously coercive for $\|\cdot\|_{\mathrm{a},h}$ with constant 1 and, since $-\Delta u = f$, the consistency error is exactly $\mathcal{E}_h(u; \cdot)$. Hence, the error estimate follows using (ii). □

Static condensation I

- Fix a basis for $\underline{U}_{h,0}^k$ with functions supported by only one T or F
- Partition the discrete unknowns into element- and interface-based:

$$\mathsf{U}_h = \begin{bmatrix} \mathsf{U}_{\mathcal{T}_h} \\ \mathsf{U}_{\mathcal{F}_h^{\rm i}} \end{bmatrix}$$

- U_h solves the following linear system:

$$\begin{bmatrix} \mathsf{A}_{\mathcal{T}_h\mathcal{T}_h} & \mathsf{A}_{\mathcal{T}_h\mathcal{F}_h^{\rm i}} \\ \mathsf{A}_{\mathcal{F}_h^{\rm i}\mathcal{T}_h} & \mathsf{A}_{\mathcal{F}_h^{\rm i}\mathcal{F}_h^{\rm i}} \end{bmatrix} \begin{bmatrix} \mathsf{U}_{\mathcal{T}_h} \\ \mathsf{U}_{\mathcal{F}_h^{\rm i}} \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}_h} \\ 0 \end{bmatrix}$$

- $\mathsf{A}_{\mathcal{T}_h\mathcal{T}_h}$ **is block-diagonal and SPD, hence inexpensive to invert**

Static condensation II

This remark suggests a two-step solution strategy:

- Element unknowns are eliminated solving the local balances

$$\mathsf{U}_{\mathcal{T}_h} = \mathsf{A}_{\mathcal{T}_h\mathcal{T}_h}^{-1}\left(\mathsf{F}_{\mathcal{T}_h} - \mathsf{A}_{\mathcal{T}_h\mathcal{F}_h^{\mathrm{i}}}\mathsf{U}_{\mathcal{F}_h^{\mathrm{i}}}\right)$$

- Face unknowns are obtained solving the global transmission problem

$$\mathsf{A}_h^{\mathrm{sc}}\mathsf{U}_{\mathcal{F}_h^{\mathrm{i}}} = -\mathsf{A}_{\mathcal{T}_h\mathcal{F}_h}^{\mathrm{T}}\mathsf{A}_{\mathcal{T}_h\mathcal{T}_h}^{-1}\mathsf{F}_{\mathcal{T}_h}$$

  with global system matrix

$$\mathsf{A}_h^{\mathrm{sc}} := \mathsf{A}_{\mathcal{F}_h\mathcal{F}_h} - \mathsf{A}_{\mathcal{T}_h\mathcal{F}_h}^{\mathrm{T}}\mathsf{A}_{\mathcal{T}_h\mathcal{T}_h}^{-1}\mathsf{A}_{\mathcal{T}_h\mathcal{F}_h}$$

  $\mathsf{A}_h^{\mathrm{sc}}$ **is SPD and its stencil involves neighbours through faces**

Numerical examples

2d test case, smooth solution, uniform refinement

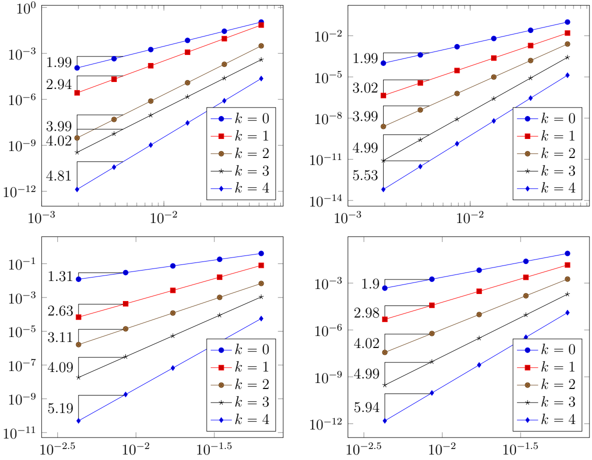


Figure: 2d test case, trigonometric solution. Energy (left) and L^2-norm (right) of the error vs. h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

Numerical examples I

3d industrial test case, adaptive refinement, cost assessment

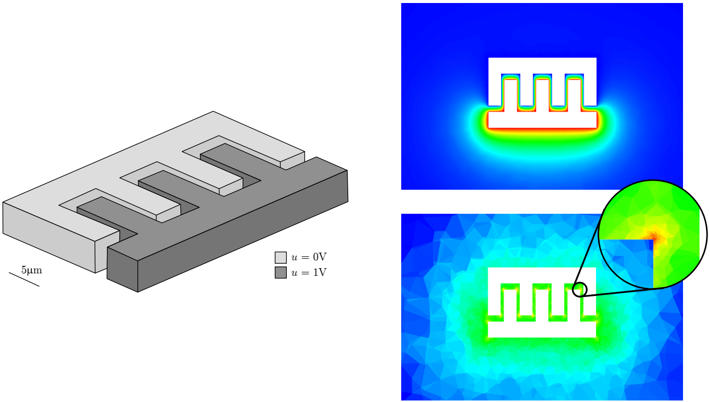
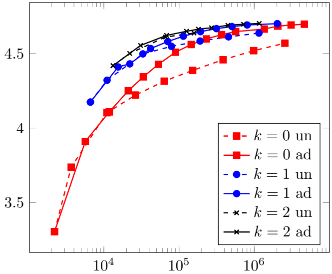


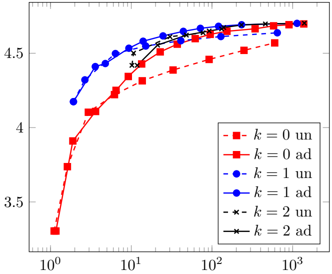
Figure: Geometry (left), numerical solution (right, top) and final adaptive mesh (right, bottom) for the comb-drive actuator test case
[Di Pietro and Specogna, 2016]

Numerical examples II

3d industrial test case, adaptive refinement, cost assessment



(a) Capacitance vs. $N_{\mathrm{dof},h}$

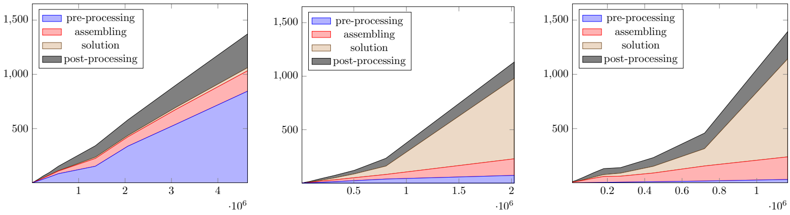


(b) Capacitance vs. computing time

Figure: Results for the comb drive benchmark.

Numerical examples III

3d industrial test case, adaptive refinement, cost assessment



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Figure: Computing wall time (s) vs. number of DOFs for the comb drive benchmark, AGMG solver.

Numerical examples I

3d test case, singular solution, adaptive coarsening

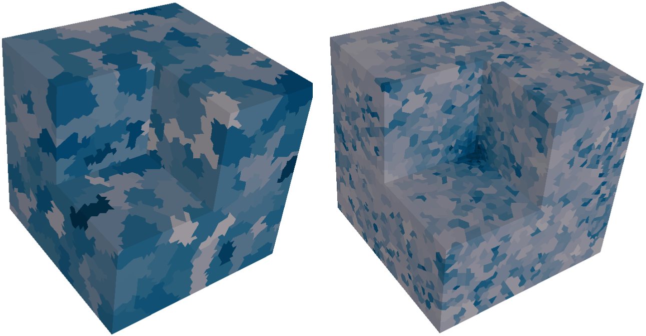
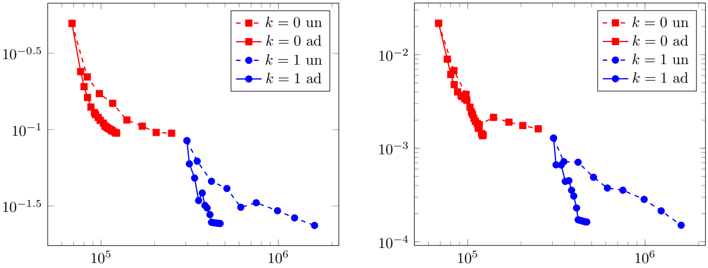


Figure: Fichera corner benchmark, adaptive mesh coarsening [Di Pietro and Specogna, 2016]

Numerical examples II

3d test case, singular solution, adaptive coarsening



(a) Energy-error vs. N_{dofs} (b) L^2-error vs. N_{ndof}

Figure: Error vs. number of DOFs for the Fichera corner benchmark, adaptively coarsened meshes

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