From physical models to advanced numerical methods through de Rham cohomology

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mainly from joint works with Jérôme Droniou

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Outline

Setting I

- Let Ω be an open connected ($b_0 = 1$) polyhedral domain of $\mathbb{R}^3$ ($b_3 = 0$)
- Assume, for the moment being, that Ω has a trivial topology, i.e.,
  - It is not crossed by any "tunnel" ($b_1 = 0$)

✗

$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$

  - It does not enclose any "void" ($b_2 = 0$)

✗

$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$

Setting II

We consider PDE models that hinge on the vector calculus operators:

$$\mathbf{grad}\, q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \ \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \ \mathrm{div}\, \boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

for smooth enough functions

$$q : \Omega \to \mathbb{R}, \qquad \boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \boldsymbol{w} : \Omega \to \mathbb{R}^3$$

Some relevant Hilbert spaces

- For simplicity, we consider problems driven by forcing terms
- To allow for physical configurations, we focus on weak formulations
- These will be based on the following Hilbert spaces:

$$
\begin{aligned}
H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \ : \ \mathbf{grad}\, q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3 \right\}, \\
\boldsymbol{H}(\mathbf{curl};\Omega) &:= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\}, \\
\boldsymbol{H}(\mathrm{div};\Omega) &:= \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega) \right\}
\end{aligned}
$$

- Given $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$, the magnetostatics problem reads:
  Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\operatorname{div};\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
\int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)
\end{aligned}
$$

- Given $\kappa > 0$ and $f \in L^2(\Omega)$, the Darcy problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} \kappa^{-1}\boldsymbol{u} - \mathbf{grad}\, p &= 0 && \text{in } \Omega, && \text{(Darcy's law)} \\ -\operatorname{div} \boldsymbol{u} &= f && \text{in } \Omega, && \text{(mass conservation)} \\ p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)} \end{aligned}$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \kappa^{-1}\boldsymbol{u} \cdot \boldsymbol{v} + \int_\Omega p \operatorname{div} \boldsymbol{v} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega), \\ -\int_\Omega \operatorname{div} \boldsymbol{u}\, q &= \int_\Omega f\, q && \forall q \in L^2(\Omega) \end{aligned}$$

A unified view

- All of the above problems are mixed formulations involving two fields
- They can be recast into the abstract setting: Find $(u, p) \in V \times Q$ s.t.

$$\begin{aligned} Au + B^\top p &= f \quad \text{in } V', \\ -Bu + Cp &= g \quad \text{in } Q' \end{aligned}$$

- Well-posedness for this problem holds under [Brezzi and Fortin, 1991]:
  - The coercivity of A in $\operatorname{Ker} B$
  - The coercivity of C in $H := \operatorname{Ker} B^\top$
  - An inf-sup condition for B: $\exists \beta \in \mathbb{R}$,

$$0 < \beta = \inf_{q \in H^\perp \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{\langle Bv, q \rangle}{\|q\|_Q \|v\|_V}$$

- **Similar properties underlie the stability of numerical approximations**

A unified tool for well-posedness: The de Rham complex

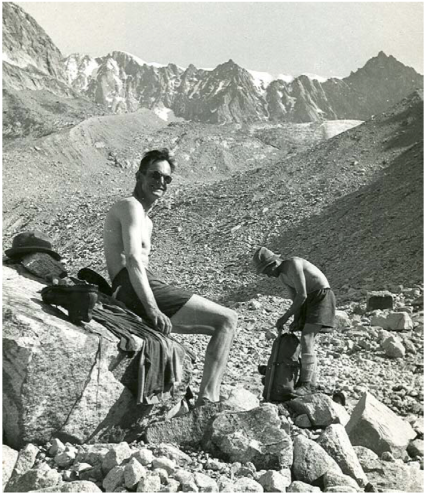

Figure: Georges de Rham (Roche 1903–Lausanne 1990)

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$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\begin{gathered}
\Omega \text{ connected } (b_0 = 1) \implies \mathrm{Ker}\,\mathbf{grad} = \mathbb{R}, \\
\mathrm{Im}\,\mathbf{grad} \subset \mathrm{Ker}\,\mathbf{curl}, \\
\mathrm{Im}\,\mathbf{curl} \subset \mathrm{Ker}\,\mathrm{div}, \\
\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(Darcy, magnetostatics)}
\end{gathered}$$

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$$\text{no “tunnels” crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im}\mathbf{grad} = \operatorname{Ker}\mathbf{curl}, \quad \text{(Stokes)}$$
$$\text{no “voids” contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im}\mathbf{curl} = \operatorname{Ker}\mathrm{div}, \quad \text{(magnetostatics)}$$
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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\operatorname{Ker} \textbf{curl} / \operatorname{Im} \textbf{grad} \quad \text{and} \quad \operatorname{Ker} \mathrm{div} / \operatorname{Im} \textbf{curl}$$

- Key consequences are Hodge decompositions and Poincaré inequalities

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- Key consequences are Hodge decompositions and Poincaré inequalities
- **Emulating these properties is key for stable discretizations**

The (trimmed) Finite Element way

Local spaces

- Let $T \subset \mathbb{R}^3$ be a tetrahedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree} \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by $\boldsymbol{x}_T$ a point inside T,

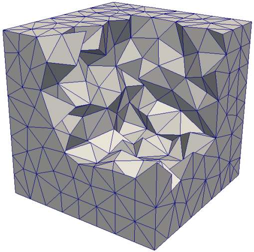
$$\begin{aligned}\mathcal{P}^k(T)^3 &= \overbrace{\mathbf{grad}\,\mathcal{P}^{k+1}(T)}^{\mathcal{G}^k(T)} \oplus \overbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathcal{P}^{k-1}(T)^3}^{\mathcal{G}^{\mathrm{c},k}(T)} \\ &= \underbrace{\mathbf{curl}\,\mathcal{P}^{k+1}(T)^3}_{\mathcal{R}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{k-1}(T)}_{\mathcal{R}^{\mathrm{c},k}(T)}\end{aligned}$$

- Define the trimmed spaces that sit between $\mathcal{P}^k(T)^3$ and $\mathcal{P}^{k+1}(T)^3$:

$$\mathcal{N}^{k+1}(T) \coloneqq \mathcal{G}^k(T) \oplus \mathcal{G}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\mathcal{RT}^{k+1}(T) \coloneqq \mathcal{R}^k(T) \oplus \mathcal{R}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

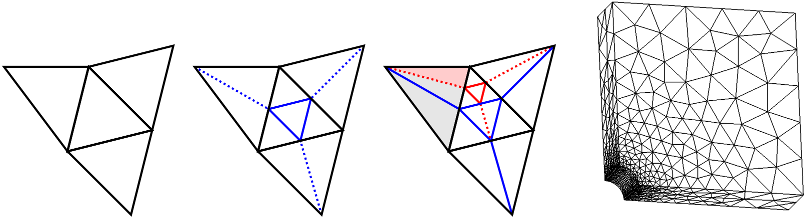
- See also [Arnold, 2018]



- Let $\mathcal{T}_h = \{T\}$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form a global FE complex:

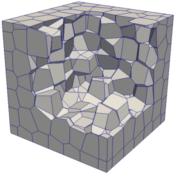
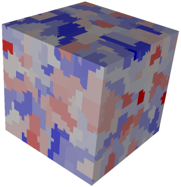
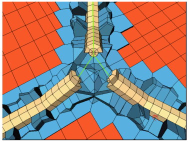
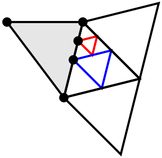
$$
\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & \mathcal{P}_c^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{N}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

- **The gluing only works on conforming meshes (simplicial complexes)!**



- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

The discrete de Rham (DDR) approach I

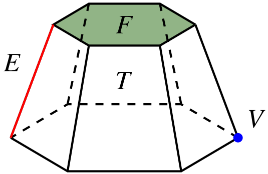
   

- **Key idea:** replace both spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{\boldsymbol{X}}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of polyhedral meshes (CW complexes) and high-order
- Key exactness and consistency properties proved at the discrete level
- Several strategies to reduce the number of unknowns on general shapes

The discrete de Rham (DDR) approach II



- DDR spaces are spanned by vectors of polynomials
- Polynomial components enable consistent reconstructions of
  - vector calculus operators
  - the corresponding scalar or vector potentials
- These reconstructions emulate integration by parts (Stokes) formulas

References

- Introduction of DDR [DP, Droniou, Rapetti, 2020]
- Present sequence and properties [DP and Droniou, 2021a]
- Application to magnetostatics [DP and Droniou, 2021b]
- Bridges with VEM [Beirão da Veiga, Dassi, DP, Droniou, 2022]
- More recent developments include:
  - Reissner–Mindlin plates [DP and Droniou, 2021c]
  - The 2D plates complex and Kirchhoff–Love plates [DP and Droniou, 2022]

$$\boldsymbol{\mathcal{RT}}^1(F) \hookrightarrow \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{sym}\,\mathbf{rot}} \boldsymbol{H}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) \xrightarrow{\mathrm{div}\,\mathbf{div}} L^2(\Omega) \xrightarrow{0} 0$$

  - The 2D Stokes complex [Hanot, 2021]

$$\mathbb{R} \hookrightarrow H^2(\Omega) \xrightarrow{\mathbf{rot}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} 0$$

  - Serendipity versions. . .
- Polyhedral analysis tools: [DP and Droniou, 2020]

Outline

- With F mesh face let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q \coloneqq (\mathbf{grad}_F\, q)^\perp \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneqq \mathrm{div}_F(\boldsymbol{v}^\perp)$$

- We derive a discrete counterpart of the 2D de Rham complex:

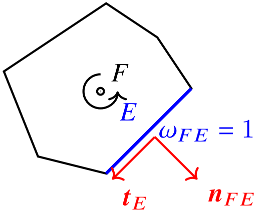
$$\mathbb{R} \hookrightarrow H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

$$\begin{aligned}\mathcal{P}^k(F)^2 &= \overbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}^{\mathcal{G}^k(F)} \oplus \overbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp \mathcal{P}^{k-1}(F)}^{\mathcal{G}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)}\end{aligned}$$

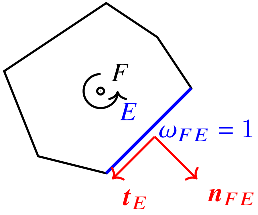
The two-dimensional case

A key remark



- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$



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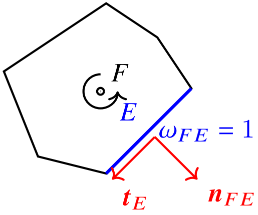


- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$

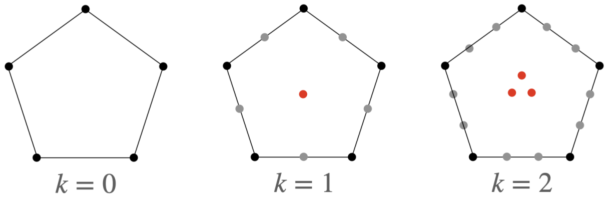
- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\underline{X}^k_{\mathbf{grad},F} := \left\{\underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_F)\right\}$$

- Let $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ be s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q := (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \;\forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \;\forall V \in \mathcal{V}_F$$



- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q \right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

The two-dimensional case

Reconstructions in $\underline{X}^k_{\mathbf{grad},F}$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

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$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- Similarly, we can reconstruct a scalar trace $\gamma^{k+1}_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^{k+1}(F)$ s.t.

$$\gamma^{k+1}_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) \coloneqq \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\ q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) q_{|E}$$

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) \coloneqq \boldsymbol{\mathcal{G}}^{k}(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$, $\forall q \in \mathcal{P}^{k}(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\ q = \int_F \boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},T}^{k-1} \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v} \cdot \boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} q_{|E}$$

The two-dimensional case

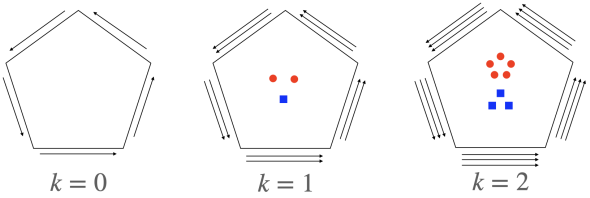
Discrete $\boldsymbol{H}(\mathrm{rot}; F)$ space

- We start from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$, $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \mathrm{rot}_F \, \boldsymbol{v} \; q = \int_F \boldsymbol{\pi}_{\mathcal{R},T}^{k-1} \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F \, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underbrace{(\boldsymbol{v} \cdot \boldsymbol{t}_E)}_{\in \mathcal{P}^k(E)} \, q_{|E}$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot}; F)$:

$$\underline{\boldsymbol{X}}_{\mathbf{curl},F}^k := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}}, (v_E)_{E \in \mathcal{E}_F} \big) : \boldsymbol{v}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$



- The face curl operator $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C_F^k \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)}$$

- Let $\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k : H^1(F)^2 \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$ collect component-wise L^2-projections
- C_F^k is polynomially consistent by construction:

$$C_F^k\big(\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k \boldsymbol{v}\big) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)$$

- Let $\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} : H^1(F)^2 \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ collect component-wise L^2-projections
- C_F^k is polynomially consistent by construction:

$$C_F^k(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v}) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F)$$

- Similarly, we can construct a tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$ s.t.

$$\boldsymbol{\gamma}^k_{\mathrm{t},F}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},F} \boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$$

- We need a discrete gradient operator from $\underline{X}^k_{\mathbf{grad},F}$ to $\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$
- To this end, let $\underline{\boldsymbol{G}}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ be s.t., $\forall \underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F \coloneqq \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), (G^k_E \underline{q}_F)_{E\in\mathcal{E}_F}\right) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$$

- If F is simply connected, the following 2D DDR complex is exact:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (face)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

- Interpolators = component-wise L^2-projections
- Discrete operators = L^2-projections of full operator reconstructions

The three-dimensional case

Local three-dimensional DDR complex and exactness

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

If the element T has a trivial topology, this complex is exact.

The three-dimensional case

Local commutation properties

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & C^\infty(\overline{T}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{T}) & \xrightarrow{0} & \{0\} \\
 & & \downarrow \underline{I}^k_{\mathrm{grad},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},T} & & \downarrow i_T & & \\
\mathbb{R} & \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} & \underline{X}^k_{\mathbf{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T) & \xrightarrow{0} & \{0\}
\end{array}
$$

- Crucial property for adjoint consistency (see below)
- Compatibility of projections with Helmholtz–Hodge decompositions

$\Longrightarrow$ Robustness of DDR numerical schemes with respect to the physics (cf. [Beirão da Veiga, Dassi, DP, Droniou, 2021], [DP and Droniou, 2022])

- Emulating integration by part formulas, we define the local potentials

$$
\begin{aligned}
P_{\mathbf{grad},T}^{k+1} &: \underline{X}_{\mathbf{grad},T}^{k} \to \mathcal{P}^{k+1}(T),\\
\boldsymbol{P}_{\mathbf{curl},T}^{k} &: \underline{\boldsymbol{X}}_{\mathbf{curl},T}^{k} \to \mathcal{P}^{k}(T)^3,\\
\boldsymbol{P}_{\mathrm{div},T}^{k} &: \underline{\boldsymbol{X}}_{\mathrm{div},T}^{k} \to \mathcal{P}^{k}(T)^3
\end{aligned}
$$

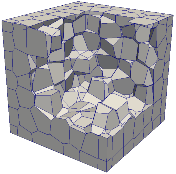
- Based on these potentials, we construct local discrete L^2-products

$$
(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}
$$

- The L^2-products are built to be polynomially exact

The three-dimensional case

Global DDR complex



$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Let $\mathcal{T}_h$ be a polyhedral mesh with elements and faces of trivial topology
- Global DDR spaces are defined gluing boundary components:

$$\underline{X}^k_{\mathbf{grad},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}^k_h : \underline{X}^k_{\mathbf{grad},h} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \ \ \underline{\boldsymbol{C}}^k_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}, \ \ D^k_h : \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathcal{P}^k(\mathcal{T}_h)$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise

Exactness of the global three-dimensional DDR complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- The global DDR complex satisfies:

$$\Omega \text{ connected } (b_0 = 1) \implies \operatorname{Im} \underline{I}^k_{\mathbf{grad},h} = \operatorname{Ker} \underline{\boldsymbol{G}}^k_h,$$
$$\text{no “tunnels” crossing } \Omega \ (b_1 = 0) \implies \operatorname{Im} \underline{\boldsymbol{G}}^k_h = \operatorname{Ker} \underline{\boldsymbol{C}}^k_h,$$
$$\text{no “voids” contained in } \Omega \ (b_2 = 0) \implies \operatorname{Im} \underline{\boldsymbol{C}}^k_h = \operatorname{Ker} D^k_h,$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im} D^k_h = \mathcal{P}^k(\mathcal{T}_h)$$

- **The latter results can be generalized to non-trivial topologies**

Exactness of the global three-dimensional DDR complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- The global DDR complex satisfies:

$$\Omega \text{ connected } (b_0 = 1) \implies \mathrm{Im}\, \underline{I}^k_{\mathbf{grad},h} = \mathrm{Ker}\, \underline{\boldsymbol{G}}^k_h,$$

$$\text{no "tunnels" crossing } \Omega\ (b_1 = 0) \implies \mathrm{Im}\, \underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\, \underline{\boldsymbol{C}}^k_h,$$

$$\text{no "voids" contained in } \Omega\ (b_2 = 0) \implies \mathrm{Im}\, \underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\, D^k_h,$$

$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\, D^k_h = \mathcal{P}^k(\mathcal{T}_h)$$

- **The latter results can be generalized to non-trivial topologies**
- We next discuss other key results focusing on magnetostatics

Discrete uniform Poincaré inequalities

- Let $(\mathrm{Ker}\,\underline{\boldsymbol{C}}_h^k)^\perp$ be the orthogonal of $\mathrm{Ker}\,\underline{\boldsymbol{C}}_h^k$ in $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ for $(\cdot,\cdot)_{\mathbf{curl},h}$. Then,

$$b_2 = 0 \implies \underline{\boldsymbol{C}}_h^k : (\mathrm{Ker}\,\underline{\boldsymbol{C}}_h^k)^\perp \to \mathrm{Ker}\,D_h^k \text{ is an isomorphism}$$

- If, moreover, $b_1 = 0$, there is $C > 0$ independent of h s.t.

$$\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} \le C\|\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \quad \forall \underline{\boldsymbol{v}}_h \in (\mathrm{Ker}\,\underline{\boldsymbol{C}}_h^k)^\perp$$

  with $\|\cdot\|_{\bullet,h}$ norm induced by $(\cdot,\cdot)_{\bullet,h}$ on $\underline{X}_{\bullet,h}^k$
- **Similar results can be proved for the gradient and the divergence**

Adjoint consistency

Adjoint consistency measures the failure to satisfy a global IBP. For the curl,

$$\int_\Omega \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v} - \int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{v} = 0 \text{ if } \boldsymbol{w} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega$$

Theorem (Adjoint consistency for the curl)

Let $\mathcal{E}_{\mathbf{curl},h} : \big(C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl};\Omega)\big) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$ *be s.t.*

$$\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h) \coloneqq (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} - \int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{P}^k_{\mathbf{curl},h} \underline{\boldsymbol{v}}_h.$$

Then, for all $\boldsymbol{w} \in C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl};\Omega)$ *s.t.* $\boldsymbol{w} \in H^{k+2}(\mathcal{T}_h)^3$*:* $\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$*,*

$$|\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h)| \le C h^{k+1} \left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \right),$$

with C *independent of* h.

Similar results can be proved for the gradient and the divergence

Outline

Discrete problem I

- With $\mu = 1$, we seek $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned}\int_\Omega \boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 &\quad \forall\boldsymbol{\tau}\in\boldsymbol{H}(\mathbf{curl};\Omega),\\ \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{A}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} &\quad \forall\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega)\end{aligned}$$

- The DDR scheme is obtained substituting

$$\boldsymbol{H}(\mathbf{curl};\Omega) \leftarrow \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \qquad \boldsymbol{H}(\mathrm{div};\Omega) \leftarrow \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$$

and

$$\begin{aligned}\int_\Omega \boldsymbol{H}\cdot\boldsymbol{\tau} &\leftarrow (\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h}, &\quad \int_\Omega \mathbf{curl}\,\boldsymbol{\tau}\cdot\boldsymbol{v} &\leftarrow (\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h},\\ \int_\Omega \mathrm{div}\,\boldsymbol{w}\;\;\mathrm{div}\,\boldsymbol{v} &\leftarrow \int_\Omega D^k_h\underline{\boldsymbol{w}}_h\;\,D^k_h\underline{\boldsymbol{v}}_h, &\quad \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v} &\leftarrow \int_\Omega \boldsymbol{J}\cdot\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{v}}_h\end{aligned}$$

Discrete problem II

- The discrete problem reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\
(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{A}}_h \, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}
\end{aligned}
$$

- Stability hinges on the exactness of the portion

$$
\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\},
$$

  which requires $b_2 = 0$

- For $b_2 \neq 0$, we need to add orthogonality to harmonic forms

Analysis I

Theorem (Stability)

Let $\Omega \subset \mathbb{R}^3$ *be an polyhedral connected domain s.t.* $b_1 = b_2 = 0$ *and set*

$$\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) \coloneqq (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h \, D_h^k \underline{\boldsymbol{v}}_h.$$

Then, it holds: $\forall (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$,

$$|||(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)|||_h \le C \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \setminus \{(\underline{\mathbf{0}}, \underline{\mathbf{0}})\}} \frac{\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h}$$

with C *independent of* h *and*

$$|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h^2 \coloneqq \|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h\|_{\mathrm{div},h}^2 + \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}^2 + \|D_h^k \underline{\boldsymbol{v}}_h\|_{L^2(\Omega)}^2.$$

Analysis II

Theorem (Error estimate for the magnetostatics problem)

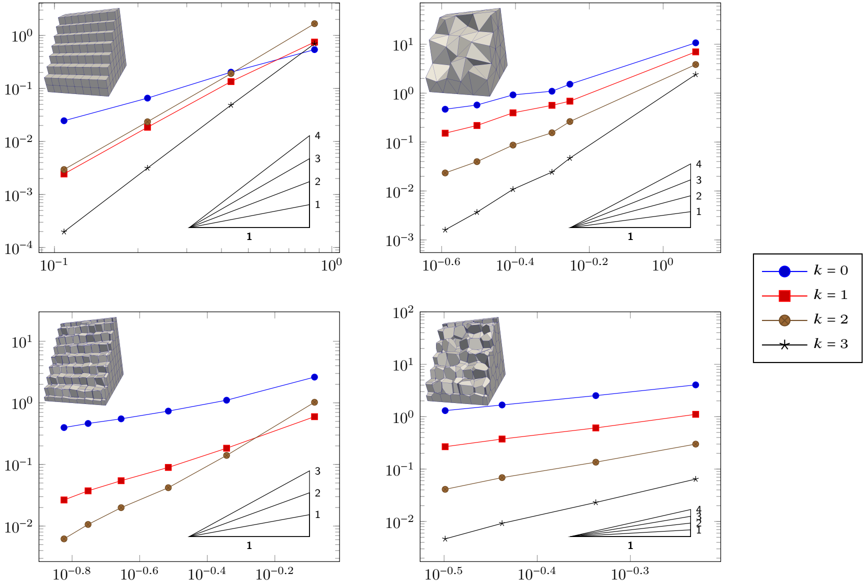
Assume $b_1 = b_2 = 0$, $\boldsymbol{H} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$, $\boldsymbol{A} \in C^0(\overline{\Omega})^3 \times H^{k+2}(\mathcal{T}_h)^3$. *Then, we have the following error estimate:*

$$\left|\!\left|\!\left|(\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{H}, \underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{A})\right|\!\right|\!\right|_h \le C h^{k+1},$$

with $C > 0$ *independent of* h.

Numerical examples

Energy error vs. meshsize



Open-source implementation available in HArDCore3D

A glance at the general case

- Let n denote the ambient dimension and Ω a polytopal set of $\mathbb{R}^n$
- For $k = 0, \ldots, n$, we define the DDR space

$$\underline{X}_{r,h}^k \coloneqq \bigtimes_{d=r}^{n} \bigtimes_{\mathsf{e} \in \mathcal{T}_{d,h}} \mathcal{P}^{k,-}\Lambda^{d-r}(\mathsf{e})$$

with $\mathcal{P}^{k,-}\Lambda^{d-r}(\mathsf{e})$ trimmed polynomial space of $(d-r)$-forms

- For $d = k+1, \ldots, n$, the discrete differential $d_{r,\mathsf{e}}^k : \underline{X}_{r,\mathsf{e}}^k \to \mathcal{P}^k\Lambda^{r+1}(\mathsf{e})$ is s.t.

$$\forall (\underline{\omega}_{\mathsf{e}}, \mu_{\mathsf{e}}) \in \underline{X}_{r,\mathsf{e}}^k \times \mathcal{P}^k\Lambda^{d-r-1}(\mathsf{e})$$
$$\int_{\mathsf{e}} d_{r,\mathsf{e}}^k \underline{\omega}_{\mathsf{e}} \wedge \mu_{\mathsf{e}} = (-1)^{k+1} \int_{\mathsf{e}} \star\omega_{\mathsf{e}} \wedge d\mu_{\mathsf{e}} + \int_{\partial\mathsf{e}} P_{r,\partial\mathsf{e}}^k \underline{\omega}_{\partial\mathsf{e}} \wedge \operatorname{tr}_{\partial\mathsf{e}} \mu_{\mathsf{e}}$$

- The discrete potential $P_{r,\mathsf{e}}^k$ is intrinsically available or defined similarly
- **Unified proofs of homological and stability properties!**

Conclusions and perspectives

- Novel approach to approximate PDEs relating to the de Rham complex
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to variable coefficients and nonlinearities

- Formalization using differential forms (ongoing work with F. Bonaldi)
- Development of novel complexes (e.g., elasticity, Hessian,. . .)
- Applications (possibly beyond continuum mechanics)

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