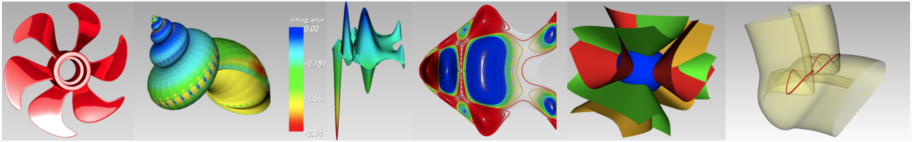
Spline functions for geometric modeling and numerical simulation

B. Mourrain
INRIA Méditerranée, Sophia Antipolis
Bernard.Mourrain@inria.fr

Montpellier, October 23-24, 2019

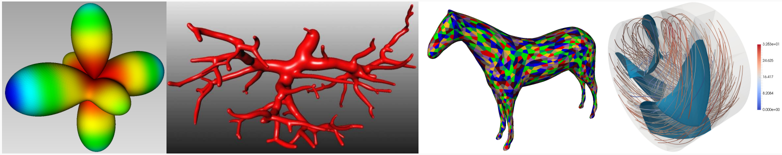
Objectives



Represent or approximate geometric objects, functions.

- ☞ High **quality** description of geometry.
- ☞ High **order** of approximation of functions.

based on piecewise polynomial models.



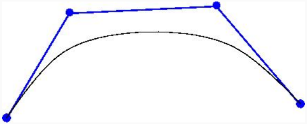
Spline functions

Univariate Bernstein representation

For any $f(x) \in \mathbb{R}[x]$ of degree d, with

$$f(x) = \sum_{i=0}^{d} c_i \binom{d}{i} (x-a)^i (b-x)^{d-i} (b-a)^{-d} = \sum_{i=0}^{d} c_i\, B_d^i(x; a, b)$$

For $c_i \in \mathbb{R}^k$, $\mathbf{c} = [c_i]_{i=0,\dots,d}$ is the *control polygon* of $f : [a,b] \to \mathbb{R}^k$.



Properties:

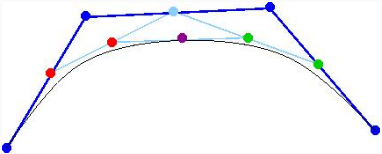
- $\sum_{i=0}^{d} B_d^i(x; a, b) = 1$; $\sum_{i=0}^{d} (a\,\frac{d-i}{d} + b\,\frac{i}{d})\, B_d^i(x; a, b) = x$;
- $f(a) = c_0, f(b) = c_d$;
- $f'(x) = d\,\sum_{i=0}^{d-1} \Delta(\mathbf{c})_i\, B_{d-1}^i(x; a, b)$ where $\Delta(\mathbf{c})_i = c_{i+1} - c_i$;
- $(x, f(x))_{x\in[a,b]} \in$ convex hull of the points $\left(a\,\frac{d-i}{d} + b\,\frac{i}{d},\ c_i\right)_{i=0..d}$
- $\#\{f(x) = 0; x \in [a,b]\} = V(\mathbf{c}) - 2p, p \in \mathbb{N}$.

De Casteljau subdivision algorithm

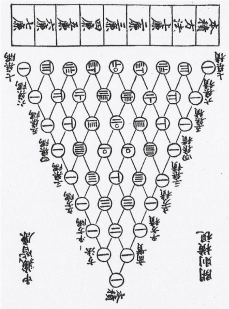
$$\begin{cases} c_i^0 = c_i, \quad i = 0, \ldots, d, \\ c_i^r(t) = \frac{b-t}{b-a}\, c_i^{r-1}(t) + \frac{t-a}{b-a}\, c_{i+1}^{r-1}(t), \quad i = 0, \ldots, d-r. \end{cases}$$

- $\mathbf{c}^-(t) = (c_0^0(t), c_0^1(t), \ldots, c_0^d(t))$ represents f on $[a, t]$.
- $\mathbf{c}^+(t) = (c_0^d(t), c_1^{d-1}(t), \ldots, c_d^0(t))$ represents f on $[t, b]$.

The geometric point of view.



The algebraic point of view.



Properties

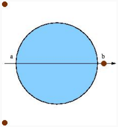
Proposition (Descartes' rule)

For $f := (\mathbf{c}, [a,b])$, $\#\{f(x) = 0; x \in [a,b]\} = V(\mathbf{c}) - 2p$, $p \in \mathbb{N}$.

Theorem

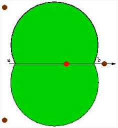
$V(\mathbf{c}^-) + V(\mathbf{c}^+) \leq V(\mathbf{c})$.

Theorem (Vincent)



If there is no complex root in the disc $D(\frac{1}{2}, \frac{1}{2}) \subset \mathbb{C}$, *then* $V(\mathbf{c}) = 0$.

Theorem (Two circles)



If there is no complex root in the union of the discs $D(\frac{1}{2} \pm \mathbf{i}\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}) \subset \mathbb{C}$ *except a simple real root, then* $V(\mathbf{c}) = 1$.

Historical notes:

Pierre Bézier (1910-1999), Renault;

Paul de Casteljau (1930-), Citroën, 1959, 1963 (secret internal reports), SMA Bézier Price 2012;

Distance between polynomials and their control polygons [1]

Let $L_i^d(t)$ be the hat function at $a+(b-a)\frac{i}{d}$.

Proposition

On the interval $[a,b]$,

$$\|\sum_i (B_i^d(t) - L_i^d(t))c_i\| \leq \frac{d\,(t-a)(b-t)}{2}\,\|\Delta^2(\mathbf{c})\|_\infty$$

$$C(d,p)(\|\Delta^2(\mathbf{c})\|_\infty - \|\Delta^3(\mathbf{c})\|_1) \leq \|\sum_i (B_i^d(t) - L_i^d(t))c_i\|_p \leq C(d,p)\,\|\Delta^2(\mathbf{c})\|_\infty$$

where $C(p,1) = \frac{d-1}{12},\ C(d,2) = \left(\frac{3d^3-5d^2+3d-1}{360d}\right)^{\frac{1}{2}}\ C(d,\infty) = \frac{d^2-\text{parity}(d)}{8d}$

☞ Quadratic convergence of the control polygon to the function (error $\times\frac{1}{4}$ when interval split at $\frac{a+b}{2}$).

[1] U. Reif, Best bounds on the approximation of polynomials andsplines by their control structure, 2000

Optimal conditioning of Bernstein basis[2]

For $\phi = (\phi_0, \ldots, \phi_d)$ a basis of $\mathbb{R}[t]_d$ and $f(t, \mathbf{c}) = \sum_{i=0}^{d} c_i \phi_i(t) \in \mathbb{R}[t]_d$,

$$|f(t, \mathbf{c} + \delta\mathbf{c}) - f(t, \mathbf{c})| = |f(t, \delta\mathbf{c})| \leq C_\phi(f, t) \, \|\delta\mathbf{c}\|_\infty$$

Partial order on bases: $\phi \preceq \psi$ if $\psi = M\phi$ with $M_{i,j} \geq 0$.

Proposition

- *If ϕ, ψ non-negative bases on $[a, b]$ with $\phi \preceq \psi$ then $C_\phi(f, t) \leq C_\psi(f, t)$ for $t \in [a, b]$.*
- *The Bernstein basis $B = (B_i^d(t; a, b))$ on $[a, b]$ is minimal for $\preceq$.*
- *If ϕ non-negative basis s.t. $((t - a)^i) \preceq \phi \preceq ((b - t)^i)$, then $\phi \sim B$.*

[2] R.T. Farouki N.T. Goodman, On the Optimal Stability of the Bernstein Basis, 1996

Piecewise polynomial functions

Knots: $t_0 \leq t_1 \leq \cdots \leq t_l \in \mathbb{R}$

Polynomials $p_0, \ldots, p_{l-1} \in \mathbb{R}[t]_d$ of degree $\leq d$ on the intervals $[t_i, t_{i+1}]$.

Regularity r_i at t_i for $i = 1, \ldots, l-1$.

$$p_i - p_{i-1} = (t - t_i)^{r_i+1}\, q_i \text{ for some } q_i \in \mathbb{R}[t]_{d-r_i-1}$$

Definition (Spline space)

For $d \in \mathbb{N}$, $\mathbf{t} = (t_0, \ldots, t_l)$, $\mathbf{r} = (r_1, \ldots, r_{l-1})$,

$$\mathcal{S}^{\mathbf{r}}_d(\mathbf{t}) = \{[p_i] \in \mathbb{R}[t]_d \mid p_i - p_{i-1} = (t - t_i)^{r_i+1}\, q_i\}$$

Dimension: $d + 1 + \sum_{i=1}^{l-1}(d - r_i)_+ \qquad (x_+ = \max\{0, x\})$

Spline basis representation

Nodes: $t_0 \leq t_1 \leq \cdots \leq t_l \in \mathbb{R}$ (repeated $d - r_i$ times at t_i).

Basis spline functions (b-spline):

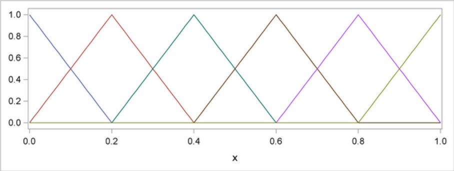
$$N_i^0(t) = \begin{cases} 1 \text{ if } t_i \leq t < t_{i+1} \\ 0 \text{ otherwise.} \end{cases}$$

$$N_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} N_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1}^{d-1}(t).$$

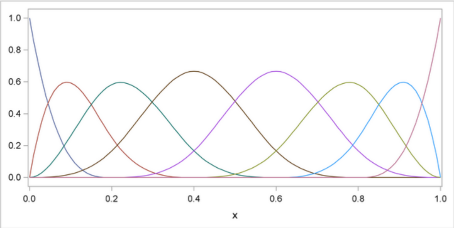
- Basis of $\mathcal{S}_d^{\mathbf{t},\mathbf{r}}$;
- Local support ($\mathrm{supp}(N_i^d) = [t_i, t_{i+d+1}]$);
- Positive functions;
- Sum to 1;

Open uniform knot vector: $t_{i+1} - t_i$ constant for $d + 1 \leq i \leq l - d - 1$.

Examples of b-spline functions



Degree: 1; Knots: $[0^2, 0.2, 0.4, 0.6, 0.8, 1^2]$; Regularity: 0



Degree: 3; Knots: $[0^4, 0.2, 0.4, 0.6, 0.8, 1^4]$; Regularity: 2

- Insertion of knot t, find the first k s.t. $t_k \leq t < t_{k+1}$ and compute:

$$\mathbf{c}_i^{(l+1)} = \frac{t_{i+d} - t}{t_{i+d} - t_i} \mathbf{c}_{i-1}^{(l)} + \frac{t - t_i}{t_{i+d} - t_i} \mathbf{c}_i^{(l)}$$

for $k - d + 1 \leq i \leq k$.

- Evaluation at t (de Boor algorithm):

$$\mathbf{c}_i^{[j+1]} = \frac{t_{i+d-j} - t}{t_{i+d-j} - t_i} \mathbf{c}_{i-1}^{[j]} + \frac{t - t_i}{t_{i+d-j} - t_i} \mathbf{c}_i^{[j]}$$

for $k - j + 1 \leq i \leq k$.

- Derivative of $f(t) = \sum_i \mathbf{c}_i N_i^d(t; \mathbf{t})$:

$$f'(t) = d \sum_i \frac{\Delta \mathbf{c}_i}{t_{d+1+i} - t_i} N_i^{d-1}(t; \mathbf{t})$$

Historical notes: Isaac J. Schoenberg (1946); Carl De Boor (1972-76); Maurice G. Cox (1972); Richard Riesenfeld (1973); Wolfgang Boehm (1980).

BS vs NURBS

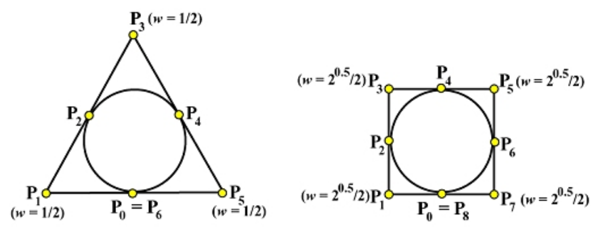
Representation of rational curves:

$$t \in [t_0, \ldots, t_l] \mapsto \frac{\sum_i c_i N_i^d(t)}{\sum \omega_i N_i^d(t)}$$

(Non-Uniform Rational B-Spline function)

Control points: $[c_i, \omega_i]$

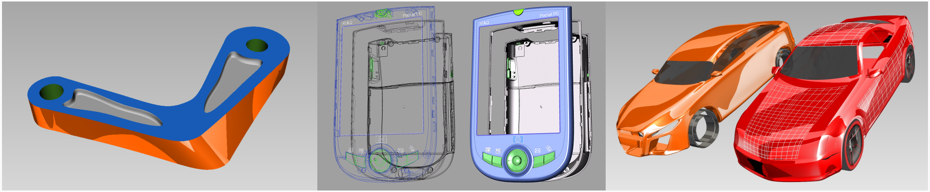
Example of a circle as a NURBS curve:



$$\frac{(1-t^2,2t)}{1+t^2} = \frac{((1-t)^2+2t(1-t), 2t(1-t)+2\,t^2)}{((1-t)^2+2t(1-t)+2\,t^2)}$$

Geometric modeling

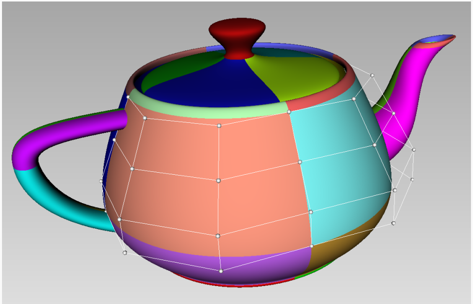
Tensor product B-splines



- Standard in Computer Aided Design (CAD);
- Define on rectangular domains;
- Grid of control points;

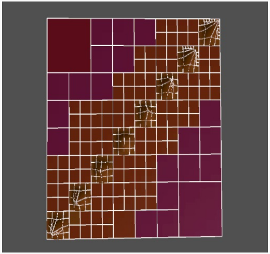
Tensor product b-spline functions:

$$(s,t) \in [s_0, s_l] \times [t_0, t_m] \mapsto \sum_i c_{i,j} N_i^{d_s}(s; \mathbf{s}) N_j^{d_t}(t; \mathbf{t})$$



- Local support of $N_{i,j}(s,t) = N_i^{d_s}(s) N_j^{d_t}(t)$ in $[s_i, s_{i+d_s+1}] \times [t_j, t_{j+d_t+1}]$
- Insertion of knots in each direction;
- Derivation formula per variable on the grid of coefficients $c_{i,j}$;

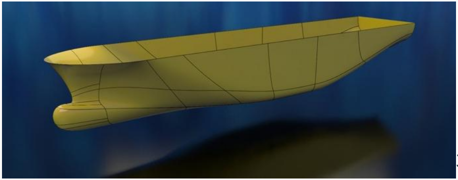
Hierarchical b-splines

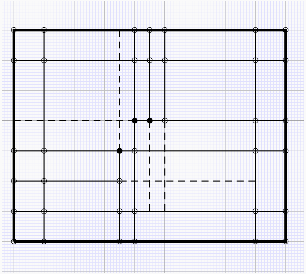


(D. Forsey, R. Bartels, 1988)

- Local refinement of the support of basis function;
- Offsets of b-spline parameterizations at different level;
- Not all possible T-mesh.

T-Splines

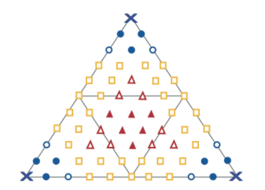

[3]

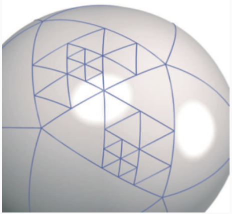


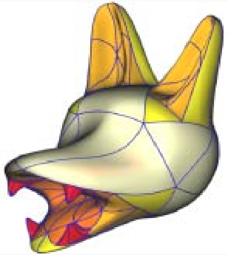
- More control for complex geometry;
- Not piecewise polynomial on the T-subdivision;
- Span by some $N(s; s_{i_0}, \ldots, s_{i_{d+2}}) \times N(t; t_{j_0}, \ldots, t_{j_{d+2}})$;
- Partition of unity with rational functions;
- Problems of linear independency;
- No characterisation of the span space.

[3] http://www.tsplines.com/

Hierarchical triangular splines







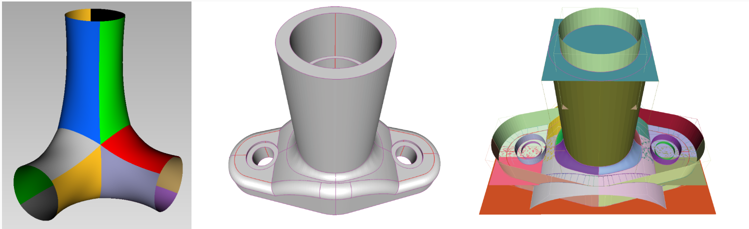
(A. Yvart, S. Hahmann, G.-P. Bonneau, 2005)

- G^1 continuity;
- Piecewise quintic polynomials;
- Arbitrary topology;

From curves to surfaces

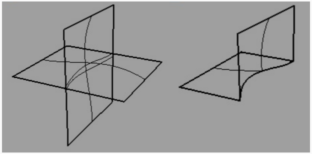
- Extrusion: $(s, t) \mapsto (C(s), t) \in \mathbb{R}^3$
- Surface of revolution: $(s, t) \mapsto (c(t) C_1(s), s(t)\, C_1(s), C_2(s))$ with $c(t)^2 + s(t)^2 = 1$
- Swept surface: $(s, t) \mapsto O(t) + M(t)\, C(s)$
- Interpolation surface: $(s, t) \mapsto \lambda_0(t)\, C_0(s) + \lambda_1(t)\, C_1(s)$ with $\lambda_0(t) + \lambda_1(t) = 1$
- ...

Multi-patch trimmed models



Geometric model made of patches, glued together along intersection curves.

Intersection of b-spline surfaces



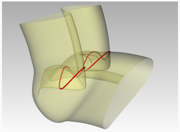
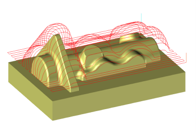
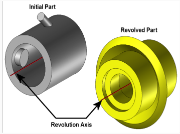
represented by b-spline curves in the parameter domains of the two surfaces and/or by their image on the two surfaces.

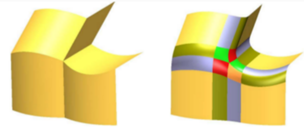
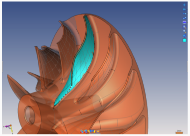
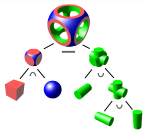
- For generic surfaces of bi-degree (d_1, d_2) and (d'_1, d'_2),
  - degree of surface $2\,d_1 d_2$, $2\,d'_1 d'_2$,
  - degree of intersection curve $4\,d_1 d_2 d'_1 d'_2$, of genus $8\,d_1 d_2 d'_1 d'_2 - 2\,d_1 d_2 (d'_1 + d'_2) - 2\,d'_1 d'_2 (d_1 + d_2) + 1$, is not rational.

- Approximate representation of the intersection curve and gaps in the models.

- Base point for rational param. $(s,t) \mapsto [\frac{f_1(s,t)}{f_0(s,t)}, \frac{f_2(s,t)}{f_0(s,t)}, \frac{f_3(s,t)}{f_0(s,t)}]$: $f_i(s_0, t_0) = 0$. Reduce the degree $2d_1 d_2 - \rho$, the genus, ...

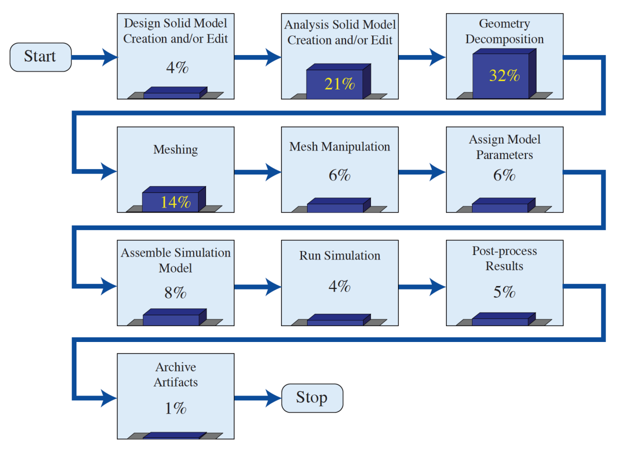
Other geometric operations

- Selfintersection 
- Offsets 
- Silhouet 

- Blending surfaces

- Reparametrisation 
- Constructive Solid Geometry (CSG)

- . . .

Isogeometric Analysis

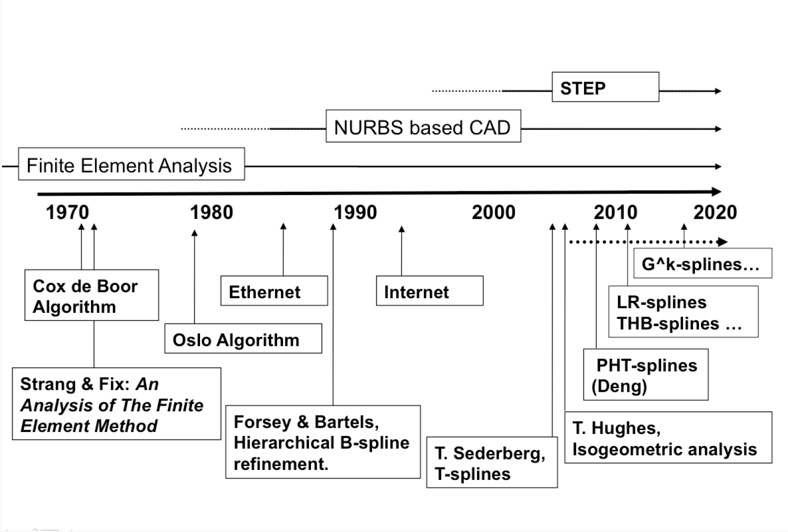
- Finite Element Analysis (FEA) developed to improve analysis in Engineering.
    - FEA was developed before the NURBS theory;
    - FEA evolution started in the 1940s and was given a rigorous mathematical foundation around 1970 (E.g, ,1973: Strang and Fix's An Analysis of The Finite Element Method)
    - An early believe that higher order representations in most cases did not contribute to better solutions
- Computer Aided Design (CAD) developed to improve the design process.
    - CAD (NURBS) and FEA evolved in different communities.
    - B-splines, 1972: DeBoor-Cox Calculation, 1980: Oslo Algorithm
    - Representation adapted to performance of earlier computers
    - Few information exchange between CAD and FEA.



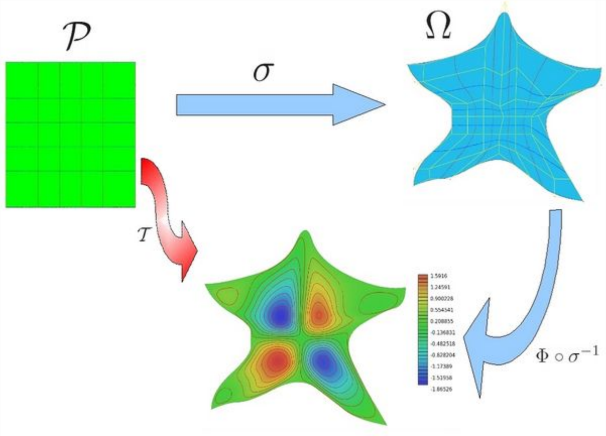
(Isogeometric Analysis: Toward Integration of CAD and FEA - J. A. Cottrell, T.J. R. Hughes, Y. Bazilevs, 2009)

☞ IsoGeometric Analysis aims at a seamless integration of Design and Analysis.

Historical perspective:



What is isogeometric analysis ?



- Choose a parametrization $\sigma : \mathcal{P} \to \Omega$ of a "computational" domain Ω.
- Use finite dimensional function space spanned by

$$\Phi_i : \mathcal{P} \quad \to \quad \mathbb{R}$$

to express the approximate solution $S : \Omega \to \mathbb{R}^d$ of a system of differential equations as

$$S(\mathbf{x}) = \left(\sum_i \lambda_i \, \Phi_i \right) \circ \sigma^{-1}(\mathbf{x}) \text{ with } \lambda_i \in \mathbb{R}^d.$$

- Pull back the solutions of the differential equations by the parameterization σ and project onto the space spanned by $\tilde{\Phi}_i(\mathbf{x}) = \Phi_i \circ \sigma^{-1}$:

$$\int_\Omega E(S)\tilde{\Phi}_i(\mathbf{x})d\mathbf{x} = \int_\mathcal{P} E(\sum_i \lambda_i \Phi_i(\mathbf{u}))\, \Phi_i(\mathbf{u})\, J_\sigma^{-1}(\mathbf{u})d\mathbf{u}$$

(Isoparametric elements: B. Irons, O. Zienkiewicz, 1968, . . . ; T. Hughes, Y. Bazilevs, . . . 2005)

Elliptic problem

Consider the following two-dimensional heat diffusion example as an illustrative model problem:

$$\begin{aligned} -\Delta u(\mathsf{x}) &= f(\mathsf{x}) && \text{in } \Omega \subset \mathbb{R}^2 \\ u(\mathsf{x}) &= g && \text{on } \partial\Omega_D \\ \partial_\nu u(\mathsf{x}) &= h && \text{on } \partial\Omega_N \end{aligned} \tag{1}$$

where

- Δ is the Laplacian operator,
- Ω is the computational domain parameterized by $\sigma : \mathcal{P} \to \Omega$,
- $u(\mathsf{x})$ is the unknown heat field,
- $f(\mathsf{x})$ is the heat source function.

Weak/variational formulation, Galerkin method

Green formula:

$$-\int_{\Omega} \Delta u\, v\, d\mathbf{x} = \int_{\Omega} \nabla u \cdot \nabla v\, d\mathbf{x} + \int_{\partial\Omega} \partial_{\nu} u\, v\, d\gamma$$

Variational formulation:

Find $u \in V$ with $u_{|\partial\Omega_D} = g$ s.t. $\forall v \in V$ with $v_{|\partial\Omega_D} = 0$,

$$a(u, v) = b(v)$$

where $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v\, d\mathbf{x}$ and $b(v) = \int_{\Omega} f\, v\, d\mathbf{x} - \int_{\partial\Omega_N} h\, v\, d\gamma$.

If $V = \langle \phi_i \rangle = \langle N_i \circ \sigma^{-1} \rangle$, $u = \sum_i c_i \phi_i$,

$$A\, \mathbf{c} = b$$

where

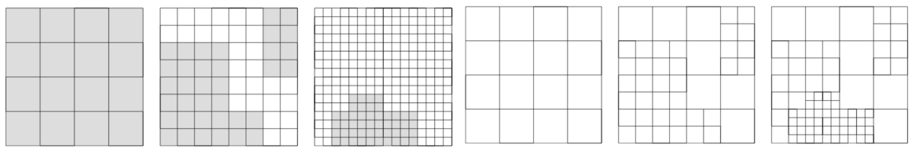
$$A_{i,j} = \int_{\Omega} \nabla\phi_i \cdot \nabla\phi_j\, d\mathbf{x} = \int_{\mathcal{P}} \nabla^t N_i J_{\sigma}^{-t} J_{\sigma}^{-1} \nabla N_j\, |J_{\sigma}|^{-1} d\mathbf{p}$$

$$b_i = \int_{\Omega} f \circ \sigma^{-1} N_i\, |J_{\sigma}|^{-1} d\mathbf{p} - \int_{\partial\Omega_N} h \circ \sigma^{-1} N_i\, |J_{\sigma}|^{-1} ds$$

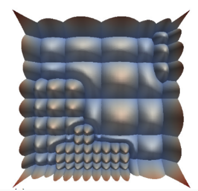
IGA with Truncated Hierarchical Bsplines (THB)[4]

Nested spaces of b-splines functions $V_0 \subset V_1 \subset \cdots \subset V_l$ with bases $\mathbf{b}_i^l(x)$.

Nested subdomains $\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_l$ and recursive subdivision



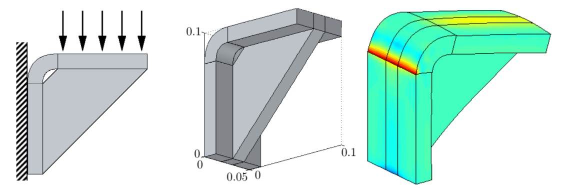
Truncated basis:

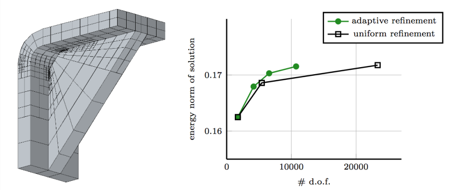


[4] THB-splines: An effective mathematical technology foradaptive refinement in geometric design andisogeometric analysis – Carlotta Giannelli, Bert Jüttler, Stefan Kleiss, Angelos Mantzaflaris, Bernd Simeon, Jaka Speh

Linear elasticity with local refinement[5]

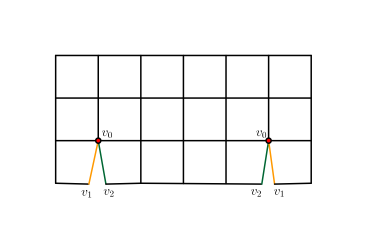
$$\sum_j \partial_j \sigma_{ij} + f_i = 0 \text{ on } \Omega;\ u_i = g_i \text{ on } \partial\Omega_{D_i};\ \sum_j \sigma_{ij} u_i = g_i \text{ on } \partial\Omega_{N_i}$$

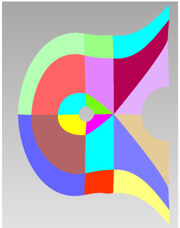




[5] THB-splines: An effective mathematical technology for adaptive refinement in geometric design andisogeometric analysis – Carlotta Giannelli, Bert Jüttler, Stefan Kleiss, Angelos Mantzaflaris, Bernd Simeon, Jaka Speh

(Singular) splines on general topology[6]





- Take a set of square faces.
- Glue them along edges.
- Choose orthogonal change of coordinates between adjacent faces.

[6] Hermite type Spline spaces over rectangular meshes with complex topological structures – Meng Wu, BM, André Galligo, Boniface Nkonga, 2017

The space $S_3^1(\mathcal{M})$ of piecewise polynomial functions on $\mathcal{M}$, which are C^1 of bi-degree $(3,3)$ is spanned by:

- for a vertex γ of valence 4: the Hermite basis functions dual to $f \to [f(\gamma), \partial_u f(\gamma), \partial_v f(\gamma), \partial_u \partial_v f(\gamma)]$.
- for a vertex γ of valence 2: the first and last Hermite basis functions with vanishing derivatives ∂_u, ∂_v at γ.
- for a vertex γ of valence $\notin \{2,4\}$: the first Hermite basis function with vanishing derivatives $\partial_u, \partial_v, \partial_u \partial_v$ at γ.

Dimension:

$$\dim \mathcal{S}_3^1(\mathcal{M}) = 4(N_b + N_0) + 2N_2 + N_3$$

where N_b is the number of boundary vertices and N_k is the number of interior basis vertices with $\deg(v) \mod 4 = k$.

Experimentation

Fixed-boundary Grad-Shafranov equation:

$$\begin{aligned} -\nabla(R(r)\nabla u) &= -g(r)f(u,r,z) \;\; \text{in} \;\; \Omega, \\ u &= 0 \;\; \text{on} \;\; \partial\Omega, \end{aligned} \tag{2}$$

where $g(r) \in L^2(\Omega)$ is a function of r and

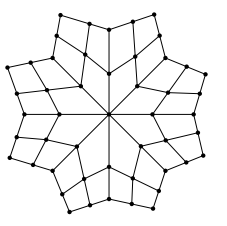
$$R(r) = \begin{pmatrix} g(r) & 0 \\ 0 & g(r) \end{pmatrix}.$$

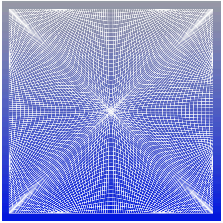
Solved iteratively the $(i+1)-$th iteration solution $u_{i+1}(r,z)$ from the solution $u_i(r,z)$:

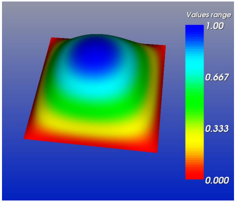
$$\begin{aligned} -\nabla(R(r)\nabla u_{i+1}(r,z)) &= -g(r)f(u_i(r,z),r,z) \;\; \text{in} \;\; \Omega, \\ u_{i+1} &= 0 \;\; \text{on} \;\; \partial\Omega, \end{aligned}$$

Elliptic boundary value problem on a square

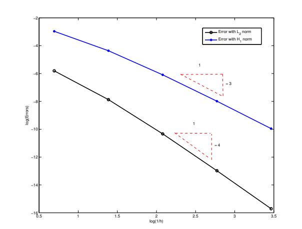
$g(r) = 1/(r+2)^2$, $f(u,r,z) = G(r,z) + u^2$ where
$G(r,z) = -(1-r^2)^2(1-z^2)^2 + 2(1-z^2) - 8(1-z^2)/(r+2) - 2(1-r^2)$.





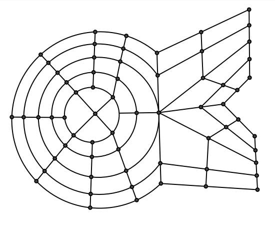


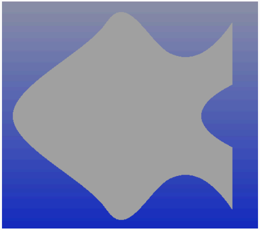
Errors with the L^2-norm and H^1-norm:



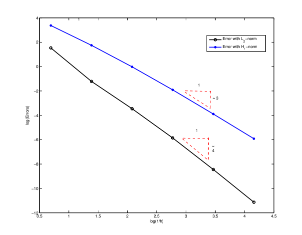
Elliptic problem on a more complex domain

$g(r) = 1$, $f = \Delta(u^*)$ with $u^* = (r+2)(r+1)\Pi_i^9 F_i(r,z)/10^4$ and $\prod_i F_i(r,z) = 0$ on $\partial\Omega$.



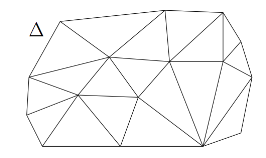


Errors with the L^2-norm and H^1-norm:



Spline spaces

Splines over a subdivision



- A decomposition of a (simply connected) domain $\mathcal{M} \subset \mathbb{R}^n$ into polygonal connected regions (cells).
- A regularity function $\mathbf{r}$ along the interior edges.

Definition

$\mathcal{S}_d^{\mathbf{r}}(\mathcal{M})$ = vector space of piecewise polynomial functions of degree $\leq d$ on each cell and of regularity $\mathbf{r}$ across the interior edges.

Problems:

- Determine its dimension;
- Compute a basis of the space $\mathcal{S}_d^{\mathbf{r}}(\mathcal{M})$, s.t.
  - the functions are positive,
  - the functions sum to one,
  - with small support,
  - reproduces $1, s, t, \dots$
  - with good power of approximation,
  - with local refinement capabilities,
  - $\dots$

One dimensional topology

Let $\mathcal{M} : t_0 \le t_1 \le \cdots \le t_l \ \in \mathbb{R}$, $\tau_i = [t_i, t_{i+1}]$, $\gamma_i = t_i$.

For each edge τ

$\mathcal{F}(\tau_i) = \mathbb{R}[u]$

$\mathcal{J}(\tau_i) = (0)$

For each vertex γ

$\mathcal{F}(\gamma_i) = \mathbb{R}[u]/\mathcal{J}(\gamma_i)$

$\mathcal{J}(\gamma_i) = ((u - t_i)^{r+1})$

$$0 \to K \to \bigoplus_{\tau \in \mathcal{M}_1^o} [\tau]\mathcal{F}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma \in \mathcal{M}_0^o} [\gamma]\mathcal{F}(\gamma) \xrightarrow{\partial_0} 0$$

with $\partial_1([\tau_i]p) = [\gamma_{i+1}]\, p - [\gamma_i]\, p$ if $[\tau_i] = [\gamma_i, \gamma_{i+1}]$ and $[\gamma_0] = [\gamma_l] = 0$.

$$\mathbf{p} = \sum_i [\tau_i] p_i \in \ker \partial_1 \qquad \text{iff} \qquad p_i - p_{i-1} \equiv 0 \mod (u - t_i)^{r+1}$$

☞ $K := \ker \partial_1 = S^r(\mathcal{M})$ and $\operatorname{im} \partial_1 = \bigoplus_{\gamma \in \mathcal{M}_0^o} \mathcal{F}(\gamma)\,[\gamma]$.

$$\dim \mathcal{S}_d^r(\mathcal{M}) - \sum_{\tau \in \mathcal{M}_1} \dim \mathcal{F}(\tau)_{[d]} + \sum_{\gamma \in \mathcal{M}_0^o} \dim \mathcal{F}(\gamma)_{[d]} = 0.$$

$\dim \mathcal{S}_d^r(\mathcal{M}) = f_1(d+1) - f_0^0(\min(r,d)+1)$ with $f_1 = |\mathcal{M}_1|$, $f_0^0 = |\mathcal{M}_0^o|$.

Two dimensional topology

- $\sigma \in \mathcal{M}_2$ set of faces of dimension 2 or cells.
- $\tau \in \mathcal{M}_1$ (resp. $\mathcal{M}_1^o$) set of (resp. interior) faces dimension 1 or edges.
- $\gamma \in \mathcal{M}_0$ (resp. $\mathcal{M}_0^o$) set of (resp. interior) faces of dimension 0 or vertices.

Definitions:

- **For** $\tau \in \mathcal{M}_1$,
  - $\ell_\tau(s,t) = 0$ be the equation of the line supporting τ.
  - $\mathfrak{I}^{\mathbf{r}}(\tau) = (\ell_\tau^{\mathbf{r}(\tau)+1})$.
- **For** $\gamma \in \mathcal{M}_0$,
  $\mathfrak{I}^{\mathbf{r}}(\gamma) = \sum_{\tau \ni \gamma} \mathfrak{I}^{\mathbf{r}}(\tau) = (\ell_\tau^{\mathbf{r}(\tau)+1})_{\tau \ni \gamma}$.

Lemma

Let $\tau \in \mathcal{M}_1$ be an edge and let $p_1, p_2 \in R$. Their derivatives coincide along τ up to order $\mathbf{r}(\tau)$ iff $p_1 - p_2 \in \mathfrak{I}^{\mathbf{r}}(\tau)$.

Topological chain complex and quotients[7]

$$
\begin{array}{lccccccc}
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
\mathfrak{I}^{\mathbf{r}}: & 0 & \to & \bigoplus_{\tau\in\mathcal{M}_{\mathbf{1}}^{\circ}}[\tau]\mathfrak{I}^{\mathbf{r}}(\tau) & \to & \bigoplus_{\gamma\in\mathcal{M}_{\mathbf{0}}^{\circ}}[\gamma]\mathfrak{I}^{\mathbf{r}}(\gamma) & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{R}: & \bigoplus_{\sigma\in\mathcal{M}_{\mathbf{2}}}[\sigma]R & \to & \bigoplus_{\tau\in\mathcal{M}_{\mathbf{1}}^{\circ}}[\tau]R & \to & \bigoplus_{\gamma\in\mathcal{M}_{\mathbf{0}}^{\circ}}[\gamma]R & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{F}^{\mathbf{r}}: & \bigoplus_{\sigma\in\mathcal{M}_{\mathbf{2}}}[\sigma]R & \to & \bigoplus_{\tau\in\mathcal{M}_{\mathbf{1}}^{\circ}}[\tau]R/\mathfrak{I}^{\mathbf{r}}(\tau) & \to & \bigoplus_{\gamma\in\mathcal{M}_{\mathbf{0}}^{\circ}}[\gamma]R/\mathfrak{I}^{\mathbf{r}}(\gamma) & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}
$$

[7] Billera, L.J. – Homology of smooth splines: generic triangulations and a conjecture of Strang, 1988; Billera, L.J., Rose, L.L. – A dimension series for multivariate splines, 1991.

- R **is the ring of polynomials in** s, t.
- $\forall \sigma \in \mathcal{M}_2$ **with its counter-clockwise boundary formed by edges** $\tau_1 = a_1 a_2, \ldots, \tau_s = a_s a_1$,

$$\partial_2([\sigma]) = [\tau_1] \oplus \cdots \oplus [\tau_s] = [a_1 a_2] \oplus \cdots \oplus [a_s a_1].$$

- $\forall \tau = \gamma_1 \gamma_2 \in \mathcal{M}_1^o$ **with** $\gamma_1, \gamma_2 \in \mathcal{M}_0$,

$$\partial_1([\tau]) = [\gamma_1] - [\gamma_2]$$

  where $[\gamma] = 0$ **if** $\gamma \notin \mathcal{M}_0^o$;
- $\forall \gamma \in \mathcal{M}_o^o$, $\partial_0([\gamma]) = 0$.
- **For** $\tau \in \mathcal{M}_1$, $\ell_\tau(s,t) = 0$ **is the equation of the line supporting** τ, $\mathfrak{I}^{\mathbf{r}}(\tau) = (\ell_\tau^{\mathbf{r}(\tau)+1})$,
- **For** $\gamma \in \mathcal{M}_0$, $\mathfrak{I}^{\mathbf{r}}(\gamma) = \sum_{\tau \ni \gamma} \mathfrak{I}^{\mathbf{r}}(\tau)$.
- The image of the map ∂_i in $\mathfrak{F}^{\mathbf{r}}$ is taken modulo $\mathfrak{I}^{\mathbf{r}}$.

Homology

Definition: $H_i(\mathfrak{C}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$.

Long exact sequence:

$$\cdots \to H_1(\mathfrak{R}) \to H_1(\mathfrak{F}^{\mathbf{r}}) \to H_0(\mathfrak{I}^{\mathbf{r}}) \to H_0(\mathfrak{R}) \to \cdots$$

Euler characteristics: for a "degree" d,

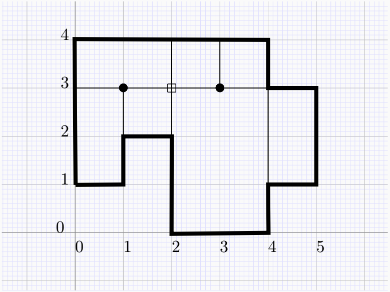
$$\sum_i (-1)^i \dim \mathfrak{F}_d^{\mathbf{r},i} = \sum_i (-1)^i \dim H_i(\mathfrak{F}_d^{\mathbf{r}})$$

Properties:

- $H_0(\mathfrak{R}) = H_1(\mathfrak{R}) = 0$
- $H_0(\mathfrak{F}^{\mathbf{r}}) = 0$
- $H_1(\mathfrak{F}^{\mathbf{r}}) = H_0(\mathfrak{I}^{\mathbf{r}})$
- $H_2(\mathfrak{F}_d^{\mathbf{r}}) = \mathcal{F}_d^{\mathbf{r}}(\mathcal{M})$

Splines on T-meshes

T-subdivision:

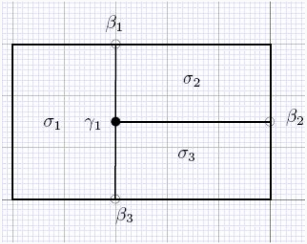


Regularity distribution: A map $\mathbf{r}$ from the horizontal and vertical nodes $\{s_1, \ldots, s_{n_1}\}$, $\{t_1, \ldots, t_{n_2}\}$ to $\mathbb{N}$, which specifies the regularity along the corresponding vertical or horizontal lines.

Spline space: Let $\mathcal{S}^{\mathbf{r}}_{m,m'}(\mathcal{M})$ be the vector space of functions which are polynomials of degree $\leqslant m$ in s, $\leqslant m'$ on each cell $\sigma \in \mathcal{M}$ and globally of class $C^{\mathbf{r}(\tau)}$ along any interior edge τ of $\mathcal{M}$.

Example

- $R = \mathbb{K}[s,t]$ polynomials in s, t, with coefficient in $\mathbb{K}$.
- $R_{m,m'}$ = polynomials of degree $\leqslant m$ in s, $\leqslant m'$ in t.

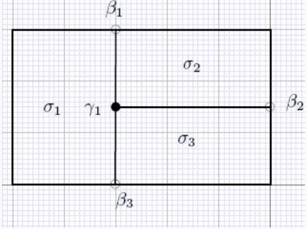


$$\mathfrak{R}_{m,m'}: \quad \bigoplus_{i=1}^{3}[\sigma_i]R_{m,m'} \xrightarrow{\partial_2} \bigoplus_{i=1}^{3}[\beta_i\gamma_1]R_{m,m'} \xrightarrow{\partial_1} [\gamma_1]R_{m,m'} \xrightarrow{\partial_0} 0$$

- $\partial_2([\sigma_1]) = [\gamma_1\beta_1] + [\beta_3\gamma_1]$, $\partial_2([\sigma_2]) = [\beta_1\gamma_1] + [\gamma_1\beta_2], \partial_2([\sigma_3]) = [\gamma_1\beta_3] + [\beta_2\gamma_1]$,
- $\partial_1([\beta_1\gamma_1]) = [\gamma_1], \partial_1([\beta_2\gamma_1]) = [\gamma_1], \partial_1([\beta_3\gamma_1] = [\gamma_1]$,
- $\partial_0([\gamma_1]) = 0$.

$$[\partial_2] = \begin{pmatrix} -I & I & 0 \\ 0 & -I & I \\ I & 0 & -I \end{pmatrix}, [\partial_1] = \begin{pmatrix} I & I & I \end{pmatrix}$$

where I is the $(m+1)(m'+1) \times (m+1)(m'+1)$ identity matrix.



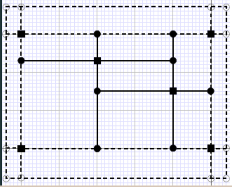
$$\mathfrak{F}^{\mathbf{r}}_{m,m'}: \quad \bigoplus_{i=\mathbf{1}}^{\mathbf{3}}[\sigma_i]R_{m,m'} \quad \to \quad \bigoplus_{i=\mathbf{1}}^{\mathbf{3}}[\beta_i\gamma_{\mathbf{1}}]R_{m,m'}/\mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_i\gamma_{\mathbf{1}}) \quad \to \quad [\gamma_{\mathbf{1}}]R_{m,m'}/\mathfrak{I}^{\mathbf{r}}_{m,m'}(\gamma_{\mathbf{1}}) \quad \to \quad 0$$

- $\mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_1\gamma_1) = \mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_3\gamma_1) = (s^{r+1}) \cap R_{m,m'}$
- $\mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_2\gamma_1) = (t^{r'+1}) \cap R_{m,m'}$
- $\mathfrak{I}^{\mathbf{r}}_{m,m'}(\gamma_1) = (s^{r+1}, t^{r'+1}) \cap R_{m,m'}$

$$[\partial_2] = \begin{pmatrix} -\Pi_1 & \Pi_1 & 0 \\ 0 & -\Pi_2 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_3 \end{pmatrix}, [\partial_1] = \begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix}$$

where Π_i (resp. P_i) is the projection matrix of $R_{m,m'}$ (resp. $R_{m,m'}/\mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_i\gamma_1)$) on $R_{m,m'}/\mathfrak{I}^{\mathbf{r}}_{m,m'}(\beta_i\gamma_1)$ (resp. $R_{m,m'}/\mathfrak{I}^{\mathbf{r}}_{m,m'}(\gamma_1)$).

Splines on planar T-meshes



- $\dim \mathcal{F}(\sigma)_{[m,m']} = (m+1)\,(m'+1)$

- $\dim \mathcal{F}(\tau)_{[m,m']} = \begin{cases} (m+1)\times(\min(r',m')+1) & \text{if } \tau \text{ is horizontal} \\ (\min(r,m)+1)\times(m'+1) & \text{if } \tau \text{ is vertical} \end{cases}$

- $\dim \mathcal{F}(\gamma)_{[m,m']} = (\min(m,\mathbf{r}(\tau_v))+1)\times(\min(\mathbf{r}(\tau_h),m')+1).$

Dimension formula

Theorem

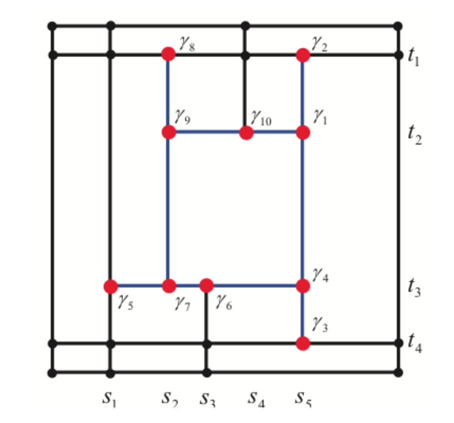
$$\begin{aligned}\dim \mathcal{F}^{\mathbf{r}}_{m,m'}(\mathcal{M}) &= (m+1)(m'+1)f_2 \\ &- (m+1)(r'+1)f_1^h - (m'+1)(r+1)f_1^v \\ &+ (r+1)(r'+1)f_0 \\ &+ h^{\mathbf{r}}_{m,m'}(\mathcal{M})\end{aligned}$$

where

- f_2 *is the number of 2-faces* $\in \mathcal{M}_2$,
- f_1^h *(resp.* f_1^v*) is the number of horizontal (resp. vertical) interior edges* $\in \mathcal{M}_1^o$,
- f_0 *is the number of interior vertices* $\in \mathcal{M}_0^o$.
- $h^{\mathbf{r}}_{m,m'}(\mathcal{M}) = \dim H_0(\mathfrak{I}^{\mathbf{r}}_{m,m'}) \geq 0$.

The bad and good news.

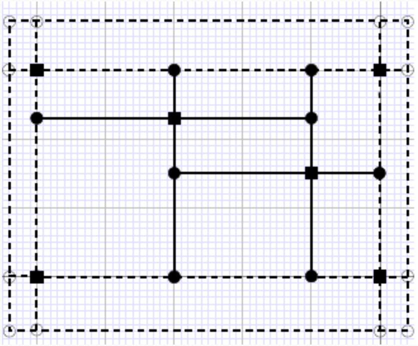
The dimension of $\mathcal{F}^{\mathbf{r}}_{m,m'}(\mathcal{M})$ may depends on the geometry:



$$0 \leq h^2_{4,4} \leq 4$$

Definitions:

- A maximal segment is a maximal union of edges of $\mathcal{M}$ that form a segment.
- It is a **maximal interior segment** if it does not intersect the boundary.
- MIS($\mathcal{M}$) is the set of maximal interior segments of $\mathcal{M}$,



Definitions:

- The maximal interior segments are **ordered** in some way: $\rho_1, \rho_2, \ldots$
- For a horizontal (resp. vertical) maximal interior segment ρ_i, $\omega(\rho_i) = \sum_{\rho \in R_i}(m + 1 - \mathbf{r}(\rho))$ (resp. $\sum_{\rho \in R_i}(m' + 1 - \mathbf{r}(\rho))$) where R_i is the set of maximal segments, which are not a maximal interior segment ρ_j of bigger index $j > i$.

Theorem

Let $\mathcal{M}$ be a hierarchical T-subdivision. Then

$$h^{\mathbf{r}}_{m,m'}(\mathcal{M}) \leqslant \sum_{\rho \in \mathrm{MIS}_h(\mathcal{M})} (m + 1 - \omega(\rho))_+ \times (m' - r') + \sum_{\rho \in \mathrm{MIS}_v(\mathcal{M})} (m - r) \times (m' + 1 - \omega(\rho))_+ .$$

Cases where $h^{\mathbf{r}}_{m,m'}(\mathcal{M}) = 0$

Corollary

If all maximal segments intersect the boundary, then $h^{\mathbf{r}}_{m,m'}(\mathcal{M}) = 0$.

Definition: a subdivision is **(k, k')-regular** for an ordering of the maximal interior segments if all the horizontal (resp. vertical) maximal interior segments are of weight $\geq k$ (resp. $\geq k'$).

Theorem

If $\mathcal{M}$ *is* $(m+1, m'+1)$*-regular. Then* $h^{\mathbf{r}}_{m,m'}(\mathcal{M}) = 0$.

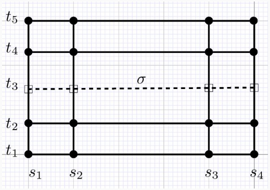
Proposition

If $m \geqslant 2r+1$ *and* $m' \geqslant 2r'+1$*, then* $h^{\mathbf{r}}_{m,m'}(\mathcal{M}) = 0$.

$$\dim \mathcal{F}^{1,1}_{2,2}(\mathcal{M}) = 9f_2 - 6f_1 + 4f_0 + h^{1,1}_{2,2}(\mathcal{M}).$$

Neighborhood: $\mathcal{N}^1(\sigma)$ is the smallest rectangle of $\mathcal{M}^\varepsilon$ that contains σ in its "interior".

Construction of 4-regular subdivisions ($h^{1,1}_{2,2}(\mathcal{M}) = 0$):



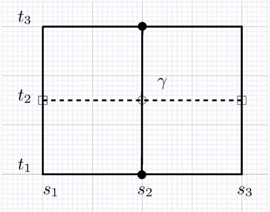
- Choose $\sigma \in \mathcal{M}_2$ and split it by an edge τ.
- Extend the edge τ on both side so that the maximal segment ρ that contains τ splits $\mathcal{N}^1(\sigma)$.

Basis functions associated to a cell σ:

$$N_\sigma(s,t) := N(s; s_{i-1}, s_{i-1}, s_i, s_i, s_{i+1})\, N(t; t_{j-1}, t_{j-1}, t_j, t_j, t_{j+1})$$

$$\dim \mathcal{C}^{1,1}_{3,3}(\mathcal{M}) = 16f_2 - 8f_1 + 4f_0 = 4(f_0^+ + f_0^b).$$

Construction of 5-regular subdivisions:



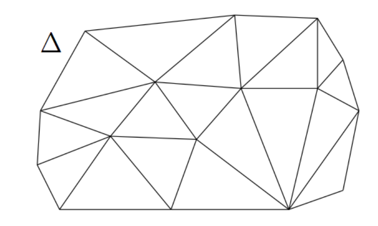
- Choose a point γ on an edge which is not a crossing vertex;
- Split the adjacent(s) cell(s) at γ.

Basis functions associated to a crossing vertex γ:

$$\begin{cases} N_\gamma^{0,0}(s,t) &= N(s; s_{i-1}, s_{i-1}, s_i, s_i, s_{i+1})\, N(t; t_{j-1}, t_{j-1}, t_j, t_j, t_{j+1}) \\ N_\gamma^{0,1}(s,t) &= N(s; s_{i-1}, s_{i-1}, s_i, s_i, s_{i+1})\, N(t; t_{j-1}, t_j, t_j, t_{j+1}, t_{j+1}) \\ N_\gamma^{1,0}(s,t) &= N(s; s_{i-1}, s_i, s_i, s_{i+1}, s_{i+1})\, N(t; t_{j-1}, t_{j-1}, t_j, t_j, t_{j+1}) \\ N_\gamma^{1,1}(s,t) &= N(s; s_{i-1}, s_i, s_i, s_{i+1}, s_{i+1})\, N(t; t_{j-1}, t_j, t_j, t_{j+1}, t_{j+1}) \end{cases}$$

Triangular splines

Triangular splines



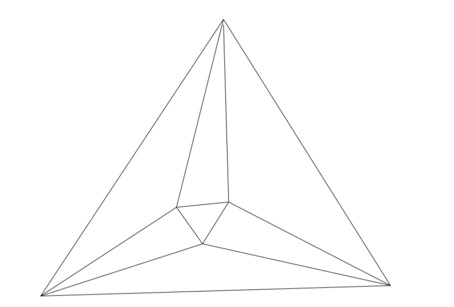
- A decomposition of a (simply connected) domain $\mathcal{M}$ into triangular cells (or polygonal regions).
- A regularity function $\mathbf{r}$ along the interior edges.

Definition

$\mathcal{S}_n^r(\mathcal{M})$ = vector space of piecewise polynomial functions of degree $\leq n$ on each cell and of regularity $\mathbf{r}$.

The bad and good news.

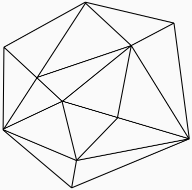
The dimension may depend on the coordinates of the vertices:



$$6 \leq \mathcal{C}_2^1(\mathcal{M}) \leq 7$$

Algebraic ingredients

For $d \in \mathbb{N}$, $\phi_{\sigma,\sigma'} = Id$,



- $\dim \mathcal{F}(\sigma)_d = \dim \mathbb{R}[u,v] = \binom{d+2}{2}$
- $\dim \mathcal{F}(\tau)_d = \dim \mathbb{R}[u,v]/(\ell^{r+1}) = \binom{d+2}{2} - \binom{d+2-(r+1)}{2}$
- For computing the dimension of $\mathcal{F}(\gamma)_d = R/(l_1^{r+1},\dots,l_t^{r+1})$, we use the resolution

$$0 \to R(-\Omega-1)^{a_i} \oplus R(-\Omega)^{b_i} \to \oplus_{j=1}^{t_i} R(-r-1) \to R \to R/\mathcal{J}(\gamma) \to 0$$

where t is the number of different slopes of the edges containing γ and $\Omega = \left\lfloor \frac{t\,r}{t-1} \right\rfloor + 1, \quad a = t\,(r+1) + (1-t)\,\Omega, \quad b = t - 1 - a.$

$$\dim \mathcal{F}(\gamma)_d = t\binom{d+2-(r+1)}{2} - b\binom{d+2-\Omega}{2} - a\binom{d+2-(\Omega+1)}{2}$$

Lower bound for splines on triangulations

Theorem

The dimension of $\mathcal{S}_d^r(\mathcal{M})$ is bounded below by

$$\dim \mathcal{S}_d^r(\mathcal{M}) \geq \binom{d+2}{2} + F_1^0 \binom{d+2-(r+1)}{2}$$
$$- \sum_{i=1}^{F_0^0} \left[t_i \binom{d+2-(r+1)}{2} - b_i \binom{d+2-\Omega_i}{2} - a_i \binom{d+2-(\Omega_i+1)}{2} \right],$$

where

- F_1^o *is the number of interior edges,*
- F_0^o *is the number of interior vertices,*
- t_i *is the number of different slopes of the edges containing the vertex* γ_i*, and*

$$\Omega_i = \left\lfloor \frac{t_i\, r}{t_i - 1} \right\rfloor + 1, \quad a_i = t_i\,(r+1) + (1-t_i)\,\Omega_i \;\text{ and }\; b_i = t_i - 1 - a_i.$$

Upper bound for splines on triangulations

Let us fix an ordering $\gamma_1, \ldots, \gamma_{f_0^0}$ for the interior vertices.

Theorem

The dimension of $S_d^r(\mathcal{M})$ is bounded by

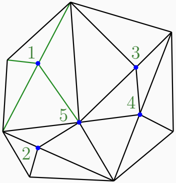
$$\dim \mathcal{S}_d^r(\mathcal{M}) \leq \binom{d+2}{2} + F_1^0 \binom{d+2-(r+1)}{2} - \sum_{i,\,\tilde{t}_i=1}^{F_0^0} \binom{d+2-(r+1)}{2}$$

$$- \sum_{i=1,\tilde{t}_i \geq 2}^{F_0^0} \left[\tilde{t}_i \binom{d+2-(r+1)}{2} - \tilde{b}_i \binom{d+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{d+2-(\tilde{\Omega}_i+1)}{2} \right],$$

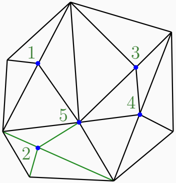
where $\tilde{t}_i$ is the number of edges with different slopes attaching the vertex γ_i to vertices on the boundary or of lower index, and

$$\tilde{\Omega}_i = \left\lfloor \frac{\tilde{t}_i\, r}{\tilde{t}_i - 1} \right\rfloor + 1, \quad \tilde{a}_i = \tilde{t}_i\,(r+1) + (1-\tilde{t}_i)\,\tilde{\Omega}_i, \quad \tilde{b}_i = \tilde{t}_i - 1 - \tilde{a}_i.$$

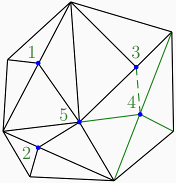
For the following numbering,



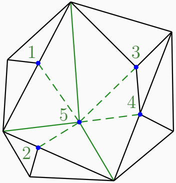
$\tilde{t}_1 = 3$



$\tilde{t}_2 = 3$



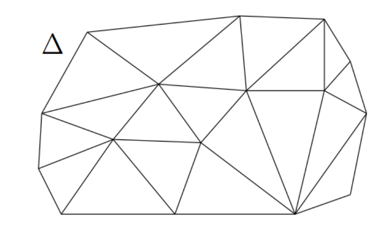
$\tilde{t}_4 = 3$

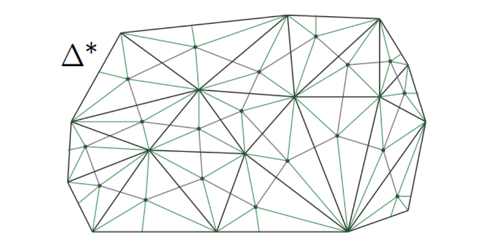


$\tilde{t}_5 = 3$

the upper bound equals the lower bound: $\dim \mathcal{S}_2^1(\mathcal{M}) = 10$.

Powell-Sabin subdivisions





(M. Powell, M. Sabin, 1977)

- Quadratic C^1, using 6 sub-triangles.
- Dimension $= 3\ V_c$ where V_c is the number of (conformal) vertices of $\mathcal{M}$.

Volumetric splines

Splines on tridimensional topological space

A similar topological complex and boundary maps:

$$0 \to \mathcal{S}_d^r(\mathcal{M}) \to \bigoplus_{\iota \in \mathcal{M}_3} \mathcal{F}(\iota) \xrightarrow{\partial_3} \bigoplus_{\sigma \in \mathcal{M}_2^0} \mathcal{F}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau \in \mathcal{M}_1^0} \mathcal{F}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma \in \mathcal{M}_0^0} \mathcal{F}(\gamma) \xrightarrow{\partial_0} 0$$

We get:

$$\dim \mathcal{S}_d^r(\mathcal{M}) = \sum_{\iota \in \mathcal{M}_3^0} \dim \mathcal{F}(\iota)_d - \sum_{\sigma \in \mathcal{M}_2^0} \dim \mathcal{F}(\sigma)_d + \boxed{\sum_{\tau \in \mathcal{M}_1^0} \dim \mathcal{F}(\tau)_d}$$
$$- \boxed{\sum_{\gamma \in \mathcal{M}_0^0} \dim \mathcal{F}(\gamma)_d} + \dim H_1(\mathcal{F})_d - \dim H_0(\mathcal{F})_d$$

- For edges τ:

$$\boxed{F(\tau) = R[u, v, w]/(\ell_1^{r+1}, \ldots, \ell_t^{r+1})}$$

as lines trough a point.

- For vertices γ, by apolarity:

$$\boxed{\dim \mathcal{F}(\gamma)_d = \dim \mathrm{R}/\langle \ell_1^{r+1}, \ldots, \ell_t^{r+1} \rangle_d = \dim(I_L^{(d-r)})_d}$$

where $I_L^{(d-r)} := \cap_{i=1}^{t} \mathfrak{m}_{\ell_i}^{d-r}$ is the *fat point ideal*.

Lower bound on $\dim \mathcal{F}(\gamma)$ from generic polynomials,
using Froberg conjecture, proved in $\mathbb{P}^2$ by D. Anick.

Upper bound in the tetrahedral case

We use:

$$\dim \mathcal{S}_d^r(\mathcal{M}) = \dim R_d + \sum_{\sigma\in\mathcal{M}_2^0} \dim \mathcal{J}(\sigma)_d - \dim \mathrm{im}\,(\partial_2)_d$$

Theorem

The dimension of $\mathcal{S}_d^r(\mathcal{M})$ is bounded above by

$$\dim \mathcal{S}_d^r(\mathcal{M}) \leq \binom{d+3}{3} + f_2^0\binom{d+3-(r+1)}{3} - \sum_{i=1}^{f_1^0}\left[\tilde{s}_i\binom{d+3-(r+1)}{3} - \tilde{b}_i\binom{d+3-\tilde{\Omega}_i}{3} - \tilde{a}_i\binom{d+3-(\tilde{\Omega}_i+1)}{3}\right]$$

with $\tilde{\Omega}_i = \lfloor \frac{\tilde{s}_i r}{\tilde{s}_i - 1}\rfloor + 1$, $\tilde{a}_i = \tilde{s}_i(r+1) + (i-\tilde{s}_i)\tilde{\Omega}_i$, *and* $\tilde{b}_i = \tilde{s}_i - 1 - \tilde{a}_i$ *if* $\tilde{s}_i > 1$, *and* $\tilde{a}_i = \tilde{b}_i = \tilde{\Omega}_i = 0$ *when* $\tilde{s}_i = 1$ *or* 0.

Lower bound on the dimension

$$F'(t,d,k)_j = \sum_i (-1)^i \dim R_{j-d\,i} \binom{t}{i}, \quad F(t,d,k) = |F'(t,d,k)|.$$

Froberg conjecture: $F(t,d,k)_j = \dim R_j/(p_1,\ldots,p_t)_j$ for generic polynomials $p_1,\ldots,p_t$ of degree d in k variables.

☞ *Lower bound for Hilbert functions of t polynomials of deg. d in k var.*

Weak Lefschetz Property: $\times\ell : M_i \to M_{i+1}$ has maximal rank $\forall i \in \mathbb{N}$.

☞ *If the WLP for I fails for $R/(L_1^{r+1},\ldots,L_t^{r+1})$ in k variables, then* $\dim R_n/(L_1^{r+1},\ldots,L_t^{r+1})_n > F(t,r+1,k)_n$.

For $k=4$, $t=5,6,7,8$, WLP fails when $r+1 \geq 3, 27, 140, 704$ (cf. H. Schenck et al).

Apolarity: $(L_1^{r+1}, \ldots, L_t^{r+1})_d^\perp = \{p \in R_d$ which vanishes with order $d - r$ "at" $L_1, \ldots L_t\}$.

For $r = d - 2$, by Alexander-Hirschowitz theorem, the dimension for generic linear forms L_i is "as expected" except for

$$(t, d, k) = (5, 4, 3), (9, 4, 4), (14, 4, 5), (7, 3, 6).$$

Segre-Harbourne-Gimigliano-Hirschowitz conjecture: dimension as expected iff there is no (-1)-special curve in the blow-up of $\mathbb{P}^2$ at $L_1, \ldots, L_t$.

Known for $t \leq 9$ [Nagata'60], $\forall t$ if $d - r \leq 12$ [Ciliberto-Miranda'98].

Lower bound in the tetrahedral case

$$\dim \mathcal{S}^r_d(\mathcal{M}) = \dim \mathcal{R}_d + \sum_{i=1}^{2} \sum_{\beta \in \mathcal{M}^{\mathbf{0}}_{\mathbf{3}-i}} (-1)^i \dim \mathcal{J}(\beta)_d + \dim \mathrm{im}\,(\partial_1)_d$$

Theorem

The dimension of $\mathcal{S}^r_d(\mathcal{M})$ is bounded below by

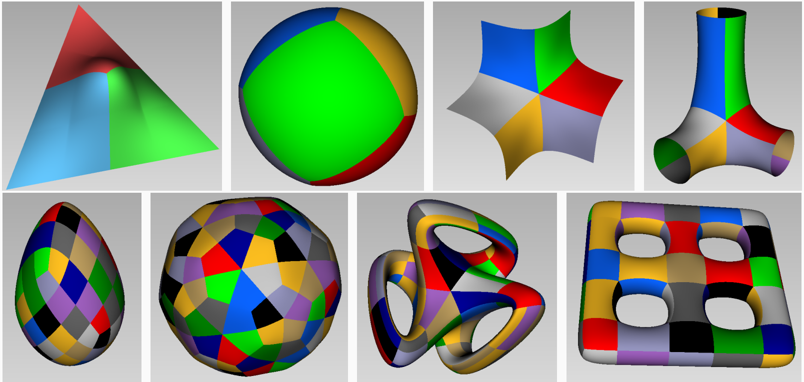
$$\dim \mathcal{S}^r_d(\mathcal{M}) \geq \binom{d+3}{3} + f^0_2 \binom{d+3-(r+1)}{3}$$
$$- \sum_{i=1}^{f^{\mathbf{0}}_{\mathbf{1}}} \left[s_i \binom{d+3-(r+1)}{3} - b_i \binom{d+3-\Omega_i}{3} - a_i \binom{d+3-(\Omega_i+1)}{3} \right]$$
$$+ f^0_0 \binom{d+3}{3} - \sum_{i=1}^{f^{\mathbf{0}}_{\mathbf{0}}} \left(\sum_{j=1}^{d} F(\zeta_i, r+1, 3)_j \right)_+$$

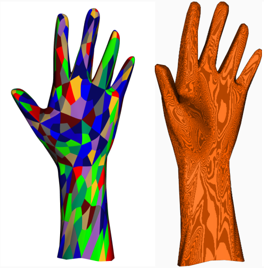
with $\Omega_i = \lfloor \frac{s_i r}{\tilde{s}_i - 1} \rfloor + 1$, $a_i = s_i(r+1) + (i - s_i)\Omega_i$, and $b_i = s_i - 1 - a_i$, and where where $F(\zeta_i, r+1, 3)$ is the Fröberg sequence for $\zeta_i = \min(3, \tilde{t}_i)$.

No time to talk about

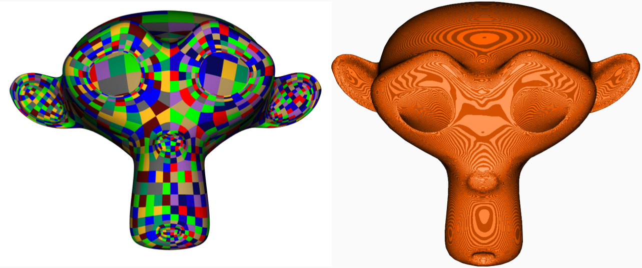
- Lower and upper bound for 3D-splines.
- Geometrically regular splines on surface of arbitrary topology.

A picture is worth a thousand words

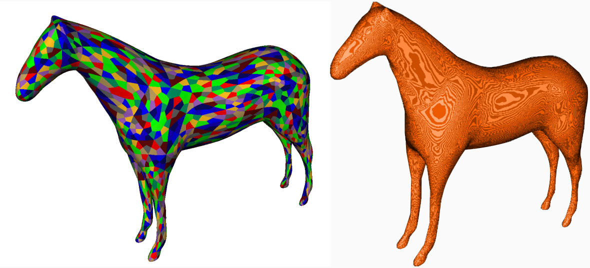




598 patches



1754 patches



G^1 Spline Surface with 3000 patches.

Problems which look for a solution

- Dimension and basis for low degree, higher regularity.
- Construction of “good” basis functions associated to vertices, edges, faces.
- Tridimensional extensions.
- Applications in fitting, isogeometric analysis.
- . . .

Thanks for your attention

- A. Blidia, B. Mourrain, N. Villamizar, *G^1-smooth splines on quad meshes with 4-split macro-patch elements*. Computer Aided Geometric Design. 2017, 52–53, pp.106-125. [hal-01491676, arXiv:1703.06717]
- B. Mourrain and N. Villamizar, *Homological techniques for the analysis of the dimension of triangular spline spaces*. Journal of Symbolic Computation 50, 564-577, 2013 [arXiv:1210.4639].
- B. Mourrain and N. Villamizar *Dimension of spline spaces on tetrahedral partitions: a homological approach*. Mathematics in Computer Sciences, 8 (2): 157–174, 2014 [arXiv:1403.0748].
- B. Mourrain, R. Vidunas and N. Villamizar *Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology*. Computer Aided Geometric Design, 45, 108–133, 2016. [hal:01196996, arXiv:1509.03274]
- B. Mourrain *On the dimension of spline spaces on planar T-meshes*. Mathematics of Computation, American Mathematical Society, 83, 847–871, 2014, [hal:00533187]