From the finite element method to the finite element exterior calculus

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Montpellier 2019

Outline

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Our guiding example

Laplace problem in mixed form…

Find $\boldsymbol{\sigma} \in \Sigma = \boldsymbol{H}(\mathrm{div};\Omega)$ and $u \in U = L^2(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau},u) = 0 & \forall\boldsymbol{\tau}\in\Sigma \\ (\operatorname{div}\boldsymbol{\sigma},v) = -(f,v) & \forall v\in U \end{cases}$$

…and its finite element approximation

Find $\boldsymbol{\sigma}_h \in \Sigma_h \subset \Sigma$ and $u_h \in U_h \subset U$ such that

$$\begin{cases} (\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau},u_h) = 0 & \forall\boldsymbol{\tau}\in\Sigma_h \\ (\operatorname{div}\boldsymbol{\sigma}_h,v) = -(f,v) & \forall v\in U_h \end{cases}$$

Standard error estimates

⟨B.–Brezzi–Fortin '13⟩

If the finite element spaces Σ_h and U_h satisfy the classical stability assumptions (inf-sup conditions) then

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)}+\|u-u_h\|_{L^2(\Omega)} \le \inf_{\boldsymbol{\tau}_h,v_h}\left(\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)}+\|u-v_h\|_{L^2(\Omega)}\right)$$

For some applications it is useful to obtain an estimate for the L^2-error $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ without assuming any extra regularity on $\mathrm{div}\,\boldsymbol{\sigma}$. This is essential for the analysis of eigenvalue problems

⟨B. '10⟩

Finite element approximation of $\boldsymbol{H}(\mathrm{div};\Omega)$

⟨B.–Brezzi–Fortin ’13⟩

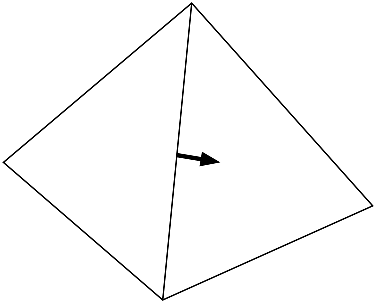
Classical finite elements for the approximation of $\boldsymbol{H}(\mathrm{div};\Omega)$ are Raviart–Thomas elements [RT] or other analogous spaces (Brezzi–Douglas–Marini [BDM], Brezzi–Douglas–Fortin–Marini [BDFM])

They are all based on the fact that a piecewise vector polynomial conforming in $\boldsymbol{H}(\mathrm{div};\Omega)$ has the normal component continuous from one element to the other

A contravariant mapping from the reference element (*Piola transform*) is used for their definitions

$$\boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} DF(\hat{\mathbf{x}})\hat{\boldsymbol{\sigma}}(\hat{\mathbf{x}})$$

Moreover by construction the spaces satisfy exactly $\mathrm{div}\,\Sigma_h = U_h$



Consequences of the Piola transform

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} \operatorname{div} \hat{\boldsymbol{\sigma}}(\hat{\mathbf{x}})$$

$$\int_T \boldsymbol{\sigma} \cdot \mathbf{grad}\, v \, d\mathbf{x} = \int_{\hat{T}} \hat{\boldsymbol{\sigma}} \cdot \mathbf{grad}\, \hat{v} \, d\hat{\mathbf{x}}$$

$$\int_T v \operatorname{div} \boldsymbol{\sigma} \, d\mathbf{x} = \int_{\hat{T}} \hat{v} \operatorname{div} \hat{\boldsymbol{\sigma}} \, d\hat{\mathbf{x}}$$

$$\int_{\partial T} \boldsymbol{\sigma} \cdot \mathbf{n} v \, d\mathbf{x} = \int_{\partial \hat{T}} \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} \hat{v} \, d\hat{\mathbf{x}}$$

Commuting diagram property

⟨Douglas–Roberts ’82⟩

A crucial remark for the analysis of the mixed Laplacian is the following commutativity:

$\Pi_\Sigma : \Sigma^+ \to \Sigma_h$
$\Pi_U : U \to U_h$

$$\boxed{\operatorname{div} \Pi_\Sigma \boldsymbol{\sigma} = \Pi_U \operatorname{div} \boldsymbol{\sigma}}$$

$$\begin{array}{ccc}
\Sigma^+ & \xrightarrow{\operatorname{div}} & U \\
\Big\downarrow \Pi_\Sigma & & \Big\downarrow \Pi_U \\
\Sigma_h & \xrightarrow{\operatorname{div}} & U_h
\end{array}$$

L^2 estimate

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) = 0 & \forall \boldsymbol{\tau} \in \Sigma_h \\ (\operatorname{div} \boldsymbol{\sigma}_h, v) = -(f, v) & \forall v \in U_h \end{cases}$$

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_\Sigma \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\ &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}) - (\operatorname{div}(\Pi_\Sigma \boldsymbol{\sigma} - \boldsymbol{\sigma}_h), u - u_h) \end{aligned}$$

It can be easily checked that $\boxed{\operatorname{div}(\Pi_\Sigma \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = 0}$

Indeed, $\operatorname{div} \boldsymbol{\sigma}_h = \Pi_U \operatorname{div} \boldsymbol{\sigma} \in U_h$ and, from the commuting diagram property, $\operatorname{div} \Pi_\Sigma \boldsymbol{\sigma} = \Pi_U \operatorname{div} \boldsymbol{\sigma} = \operatorname{div} \boldsymbol{\sigma}_h$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}) \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|\boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}\|_0$$

$$\boxed{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le \|\boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}\|_0}$$

Outline

Mixed approximation of Laplace eigenproblem

Find $\lambda \in \mathbb{R}$ and $u \in L^2(0,\pi)$ such that for some $s \in H^1(0,\pi)$

$$\begin{cases} (s,t) + (t',u) = 0 & \forall t \in H^1(0,\pi) \quad s = u' \\ (s',v) = -\lambda(u,v) & \forall v \in L^2(0,\pi) \quad s' = -\lambda u \end{cases}$$

Exact solution

$\lambda_k = k^2$, $(u_k, s_k) = (\sin(kx), k\cos(kx))$

After conforming discretization $\Sigma_h \subset \Sigma = H^1(0,\pi)$ and $U_h \subset U = L^2(0,\pi)$ the discrete problem has the following matrix form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The good element

$P_1 - P_0$ scheme (in general, $P_{\ell+1} - P_\ell$)

Same eigenvalues as for the standard Galerkin P_1 scheme

$$\lambda_h^{(k)} = \frac{6}{h^2} \cdot \frac{1 - \cos kh}{2 + \cos kh} \qquad (\to k^2)$$

$$u_h^{(k)}|_{]ih,(i+1)h[} = \frac{s_h^{(k)}(ih) - s_h^{(k)}((i+1)h)}{h\lambda_h^{(k)}}$$

$$s_h^{(k)}(ih) = k\cos(kih)$$

$i = 0, \ldots, N \quad (N = \text{number of intervals})$

$k = 1, \ldots, N$

Commuting diagram for the good element

$$\begin{array}{ccc} H^1(0,\pi) & \xrightarrow{d/dx} & L^2(0,\pi) \\ \Big\downarrow \Pi_1 & & \Big\downarrow \Pi_0 \\ P_1 & \xrightarrow[d/dx]{} & P_0 \end{array}$$

Consequence of $$\frac{\int_a^b v'(x)\,dx}{\int_a^b 1\,dx} = \frac{v(b)-v(a)}{b-a}$$

$$\boxed{\Pi_0(v') = (\Pi_1 v)'}$$

A troublesome element

$P_1 - P_1$ scheme

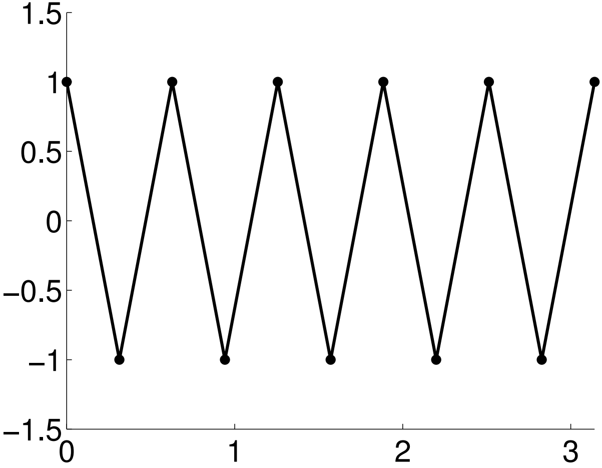
	Computed eigenvalue (rate)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
0	0.0000	-0.0000	-0.0000	-0.0000	-0.0000
1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	7.4257	8.5541 (1.8)	8.8854 (2.0)	8.9711 (2.0)	8.9928 (2.0)
9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
36	38.7154	29.7390 (-1.2)	34.2165 (1.8)	35.5415 (2.0)	35.8846 (2.0)
36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

Remark

The eigenvalues are not always approximated from above

First spurious eigenfunction

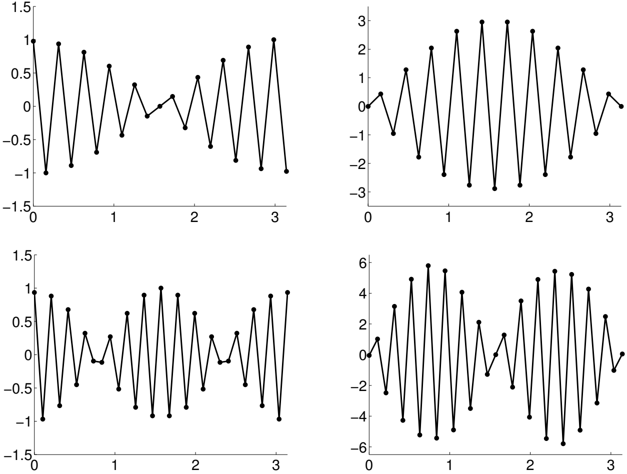
$\lambda = 0$



Higher order spurious eigenfunctions

$\lambda \simeq 9$

$\lambda \simeq 36$

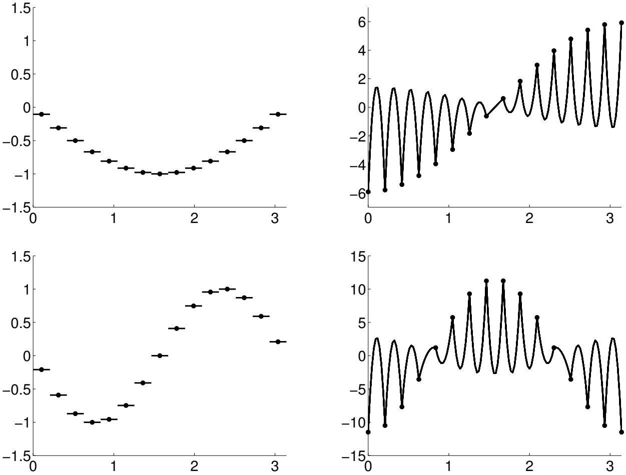


An intriguing element

$P_2 - P_0$ scheme

	Computed eigenvalue (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
#	8	16	32	64	128

Eigenfunctions for the $P_2 - P_0$ element



Another intriguing example in 2D

Neumann eigenvalue problem for the Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in L^2_0(\Omega)$ such that for some $\boldsymbol{\sigma} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u,v) & \forall v \in L^2_0(\Omega) \end{cases}$$

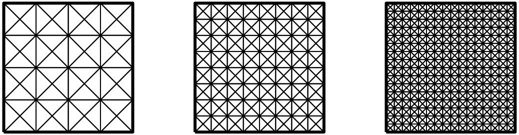
Exact solution

$\lambda_{m,n} = m^2 + n^2$

$u_{m,n} = \cos(mx)\cos(ny)$

$\boldsymbol{\sigma}_{m,n} = -(m\sin(mx)\cos(ny), n\cos(mx)\sin(ny))$

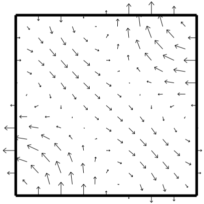
Criss-cross mesh sequence, $P_1 - \mathrm{div}(P_1)$ scheme



	Computed eigenvalue (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
dof	11	47	191	767	3071

Spurious modes

7



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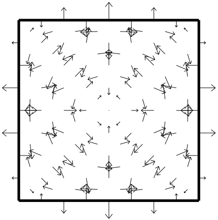

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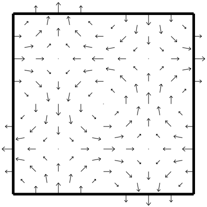
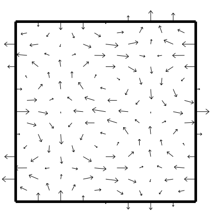


More spurious modes

15



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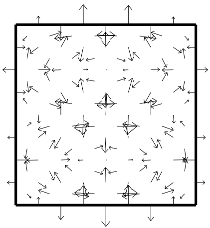

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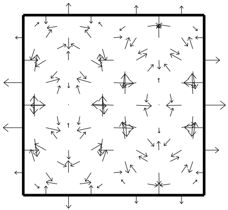

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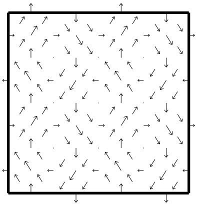


Source vs. eigenvalue problem in mixed form

⟨B.–Brezzi–Gastaldi '00⟩

N.B.
The criss-cross $P_1 - \mathrm{div}(P_1)$ element is a good element for the *source* problem (inf-sup condition OK!)

The $Q_1 - P_0$ scheme

⟨B.–Durán–Gastaldi '99⟩

The discrete eigenvalues can be explicitly computed:

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\boldsymbol{\sigma}_h^{(mn)} = (\sigma_1^{(mn)}, \sigma_2^{(mn)})$$

$$\sigma_1^{(mn)}(x_i, y_j) = \frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j)$$
$$\sigma_2^{(mn)}(x_i, y_j) = \frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

Does it converge?

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\downarrow$$

$$m^2 + n^2$$

(as h goes to 0)

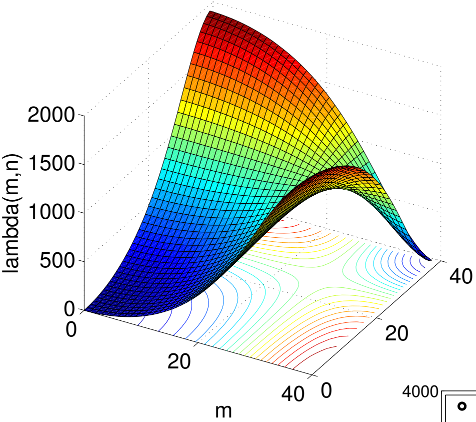
	Computed eigenvalue (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (-2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (-0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)

Wrong proof?

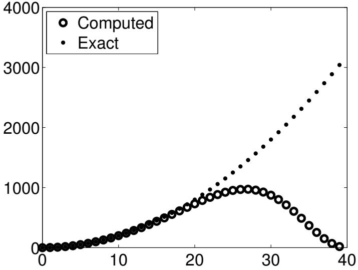
$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

Indeed, if $h = \pi/N$, we have:

$$\lim_{N\to\infty} \lambda_h^{(N-1,N-1)} = 18$$



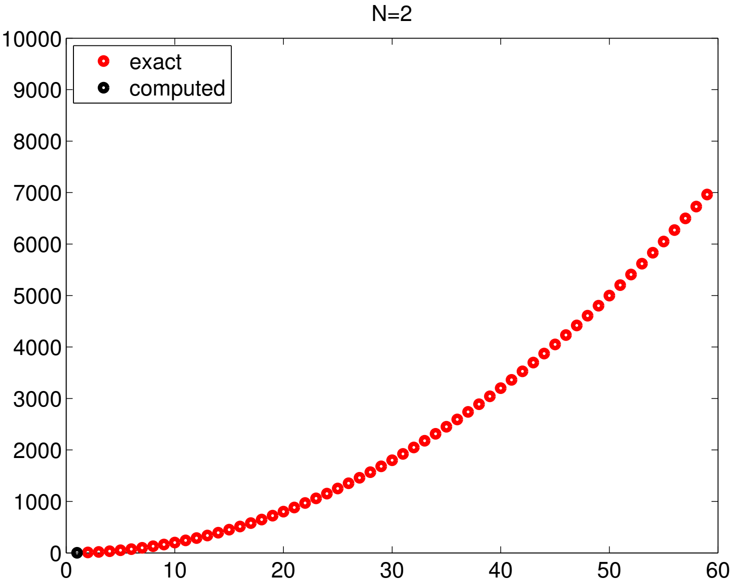
2000
1500
1000
500
0
lambda(m,n)
0
20
40
m
0
20
40



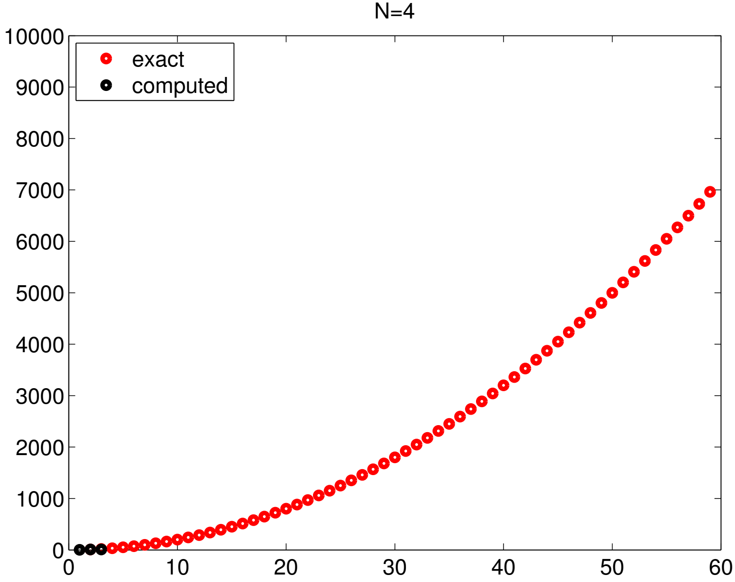
Computed
Exact
4000
3000
2000
1000
0
0
10
20
30
40

Plot for $m = n$

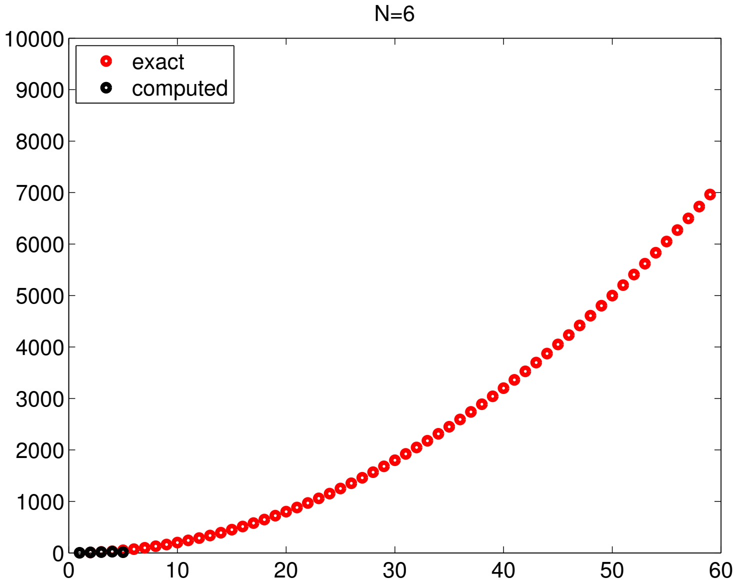
Pointwise vs. uniform convergence



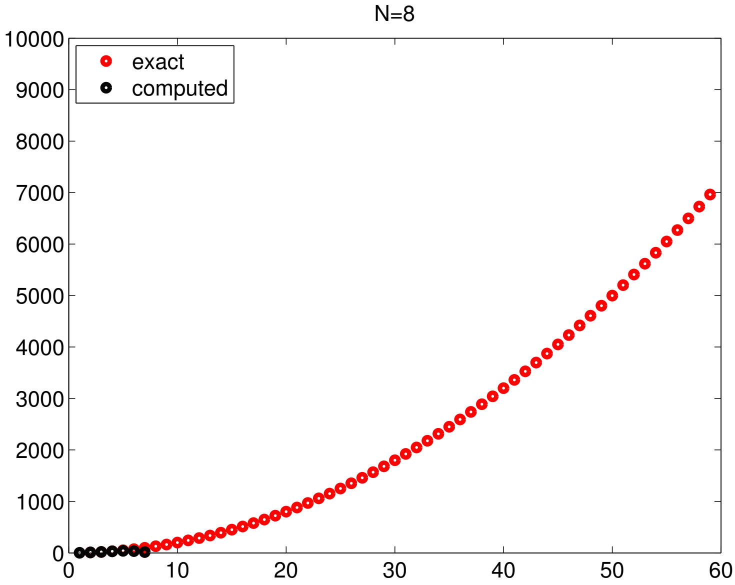
Pointwise vs. uniform convergence



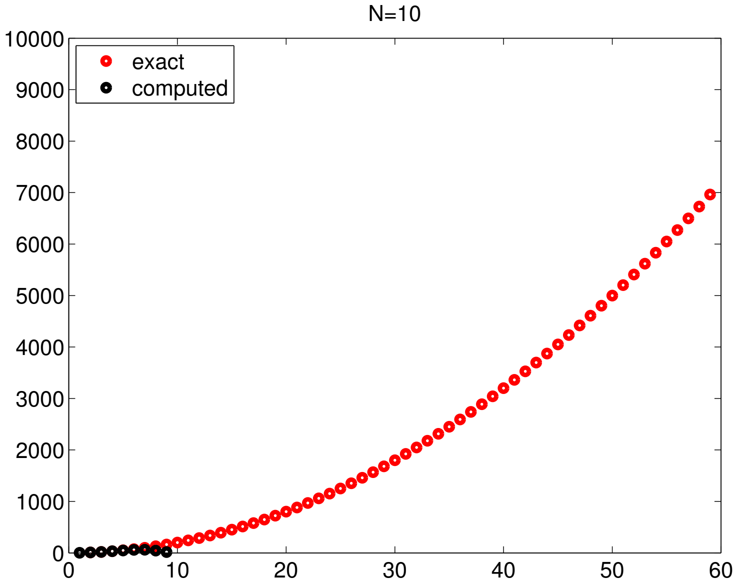
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



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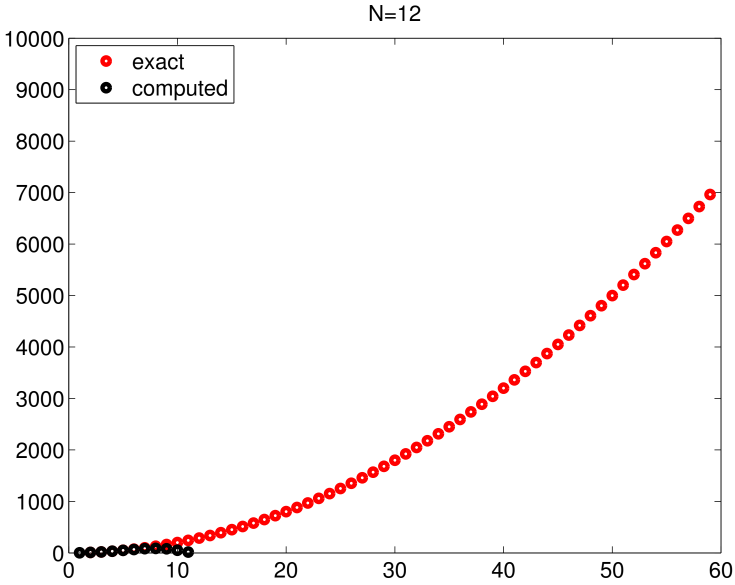

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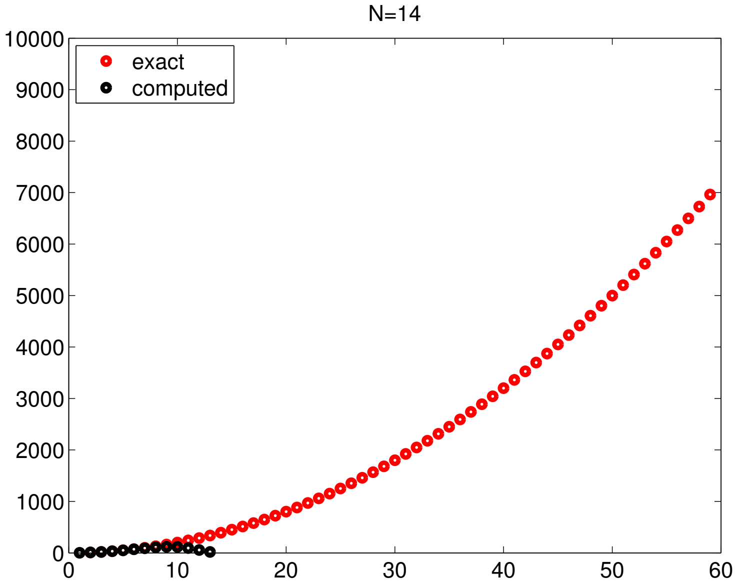

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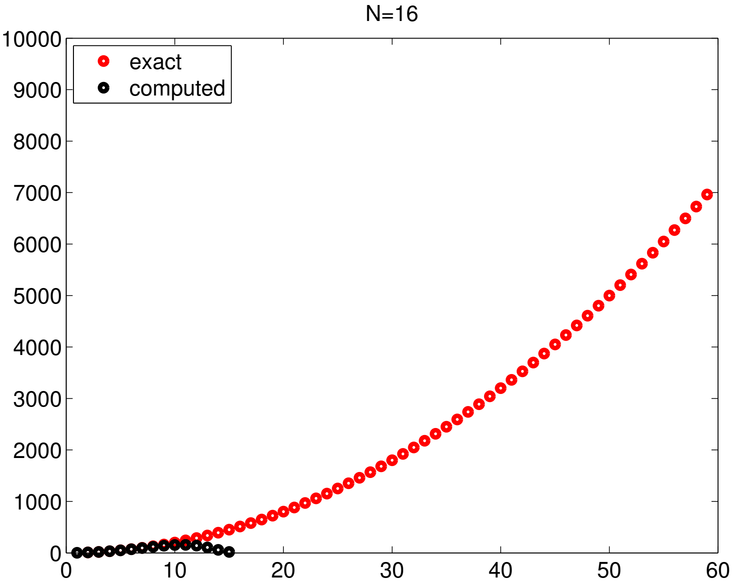

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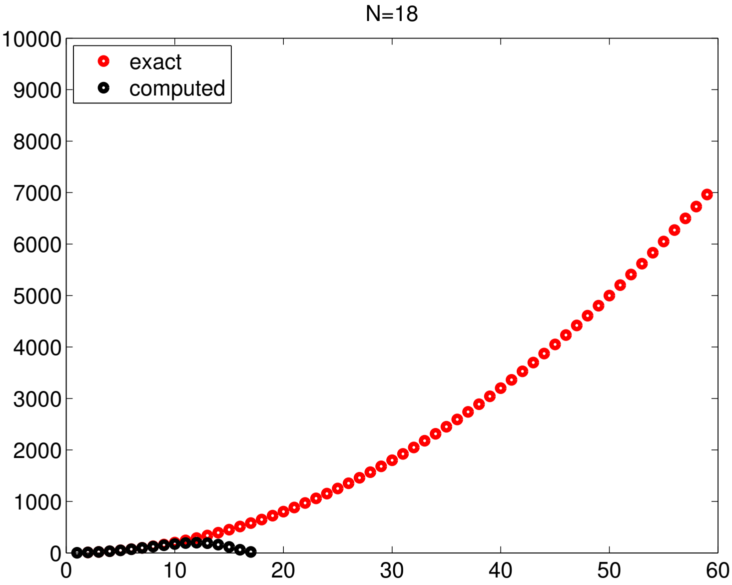

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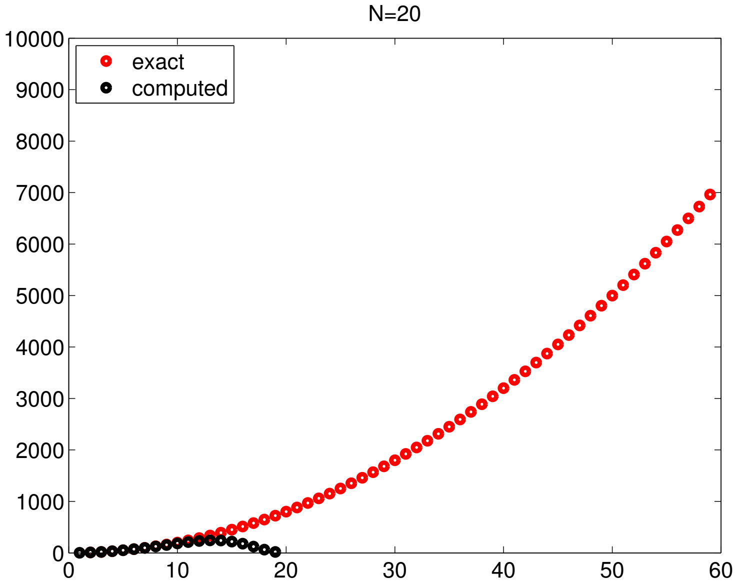

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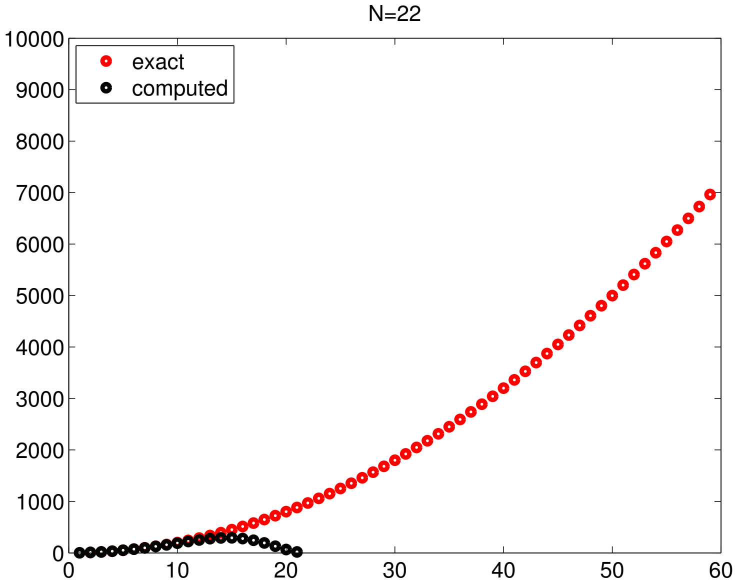

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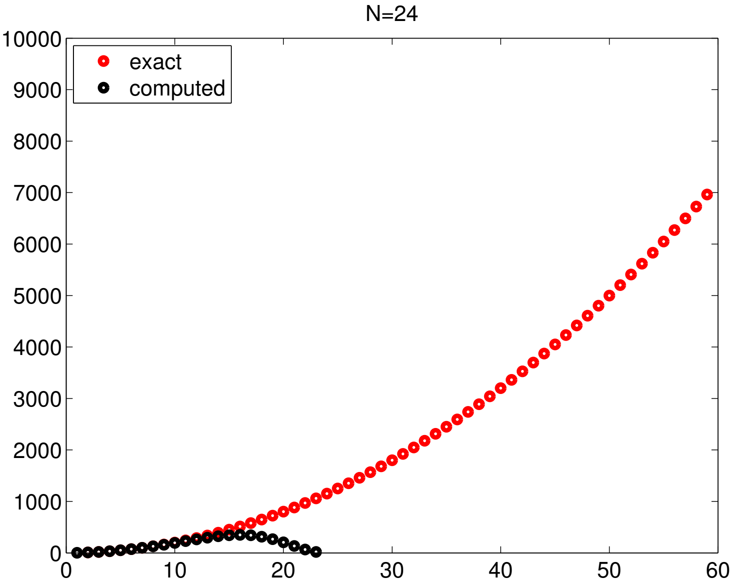

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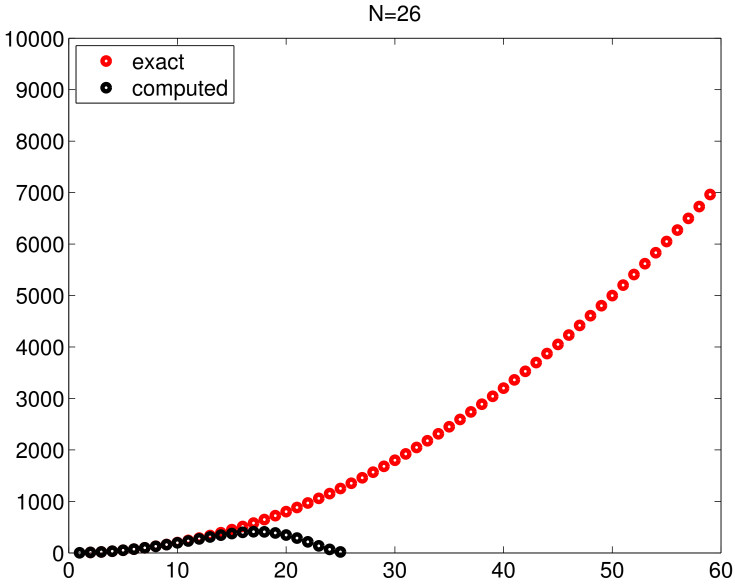

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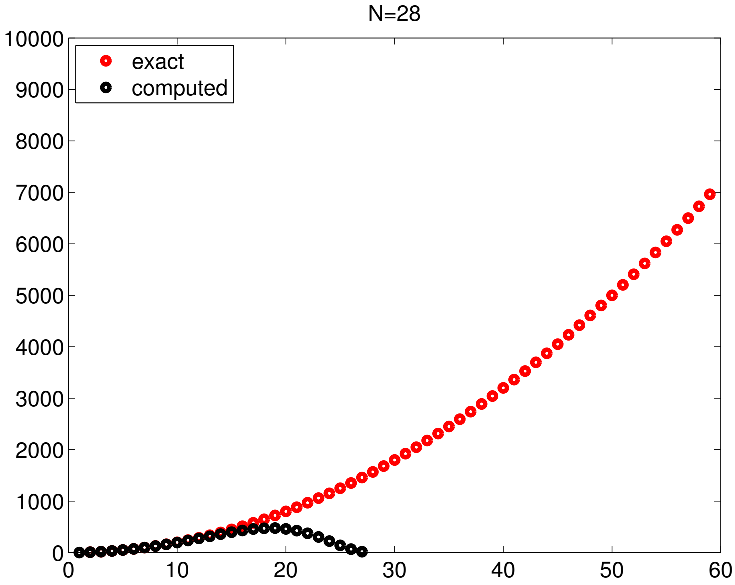

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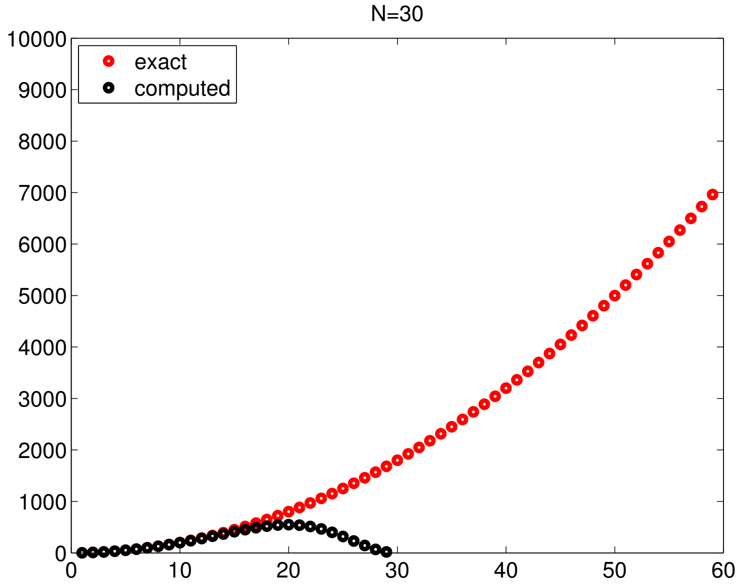

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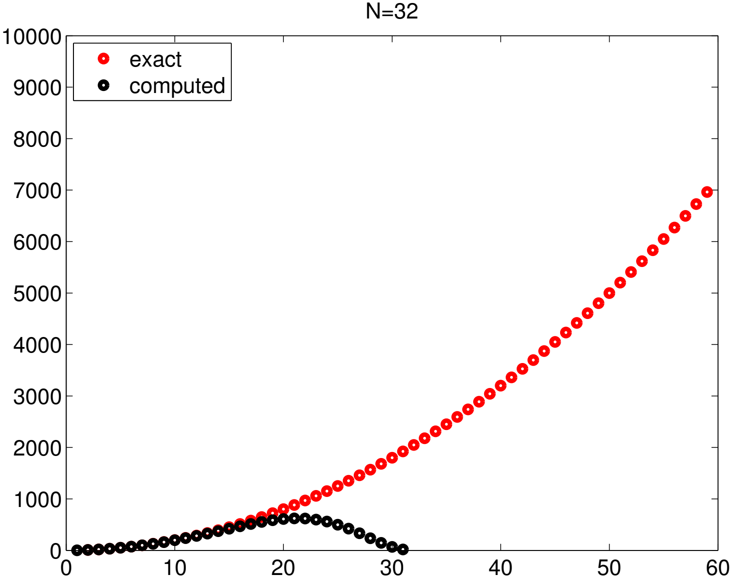

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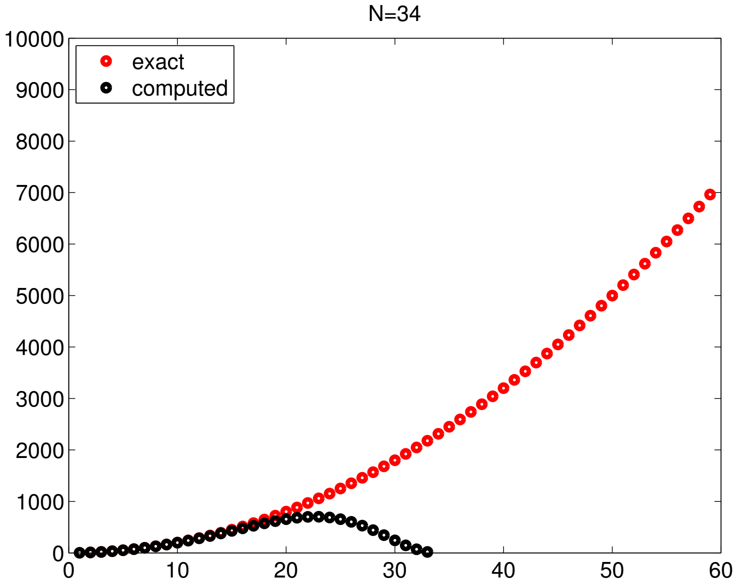

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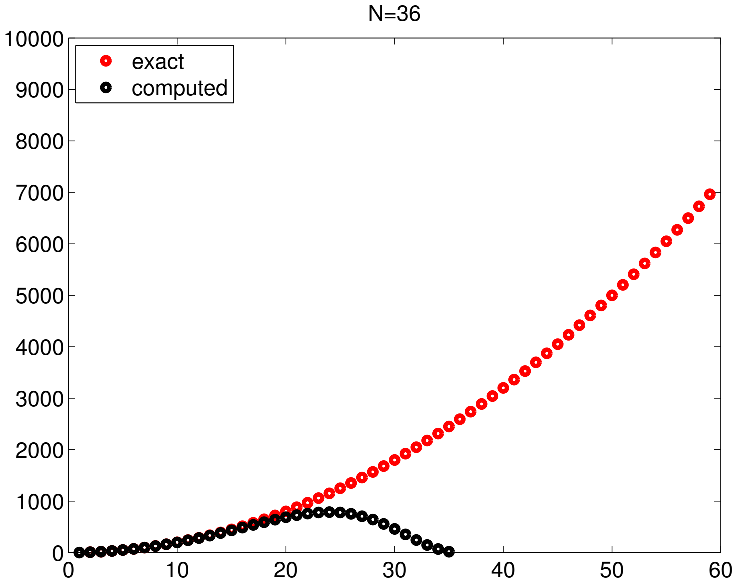

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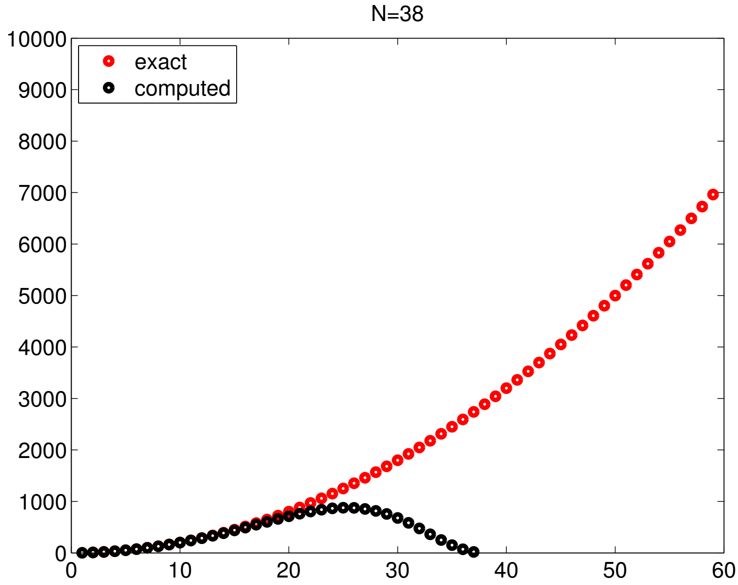

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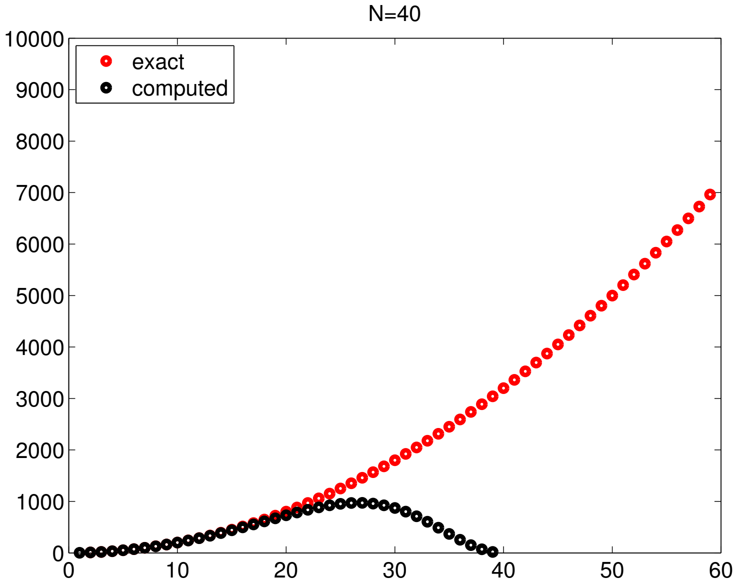

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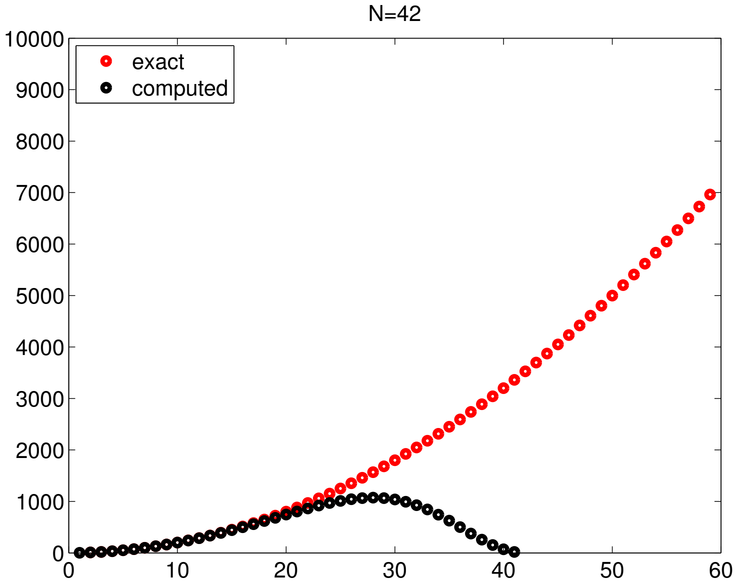

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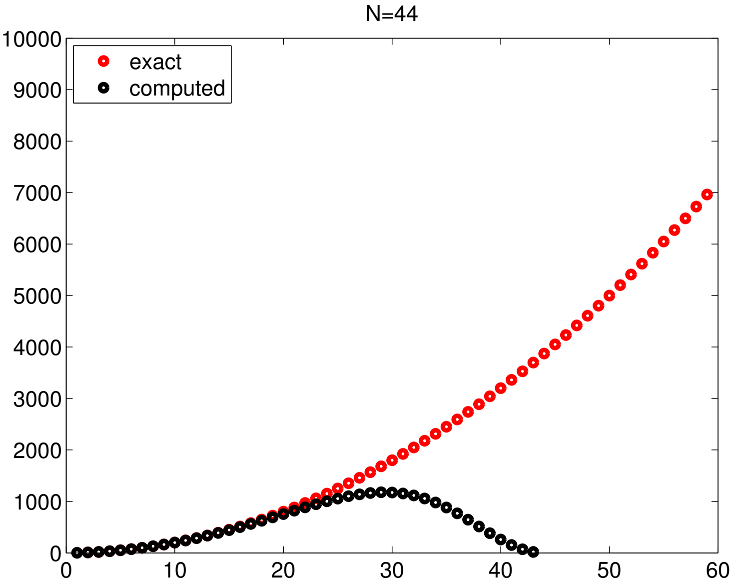

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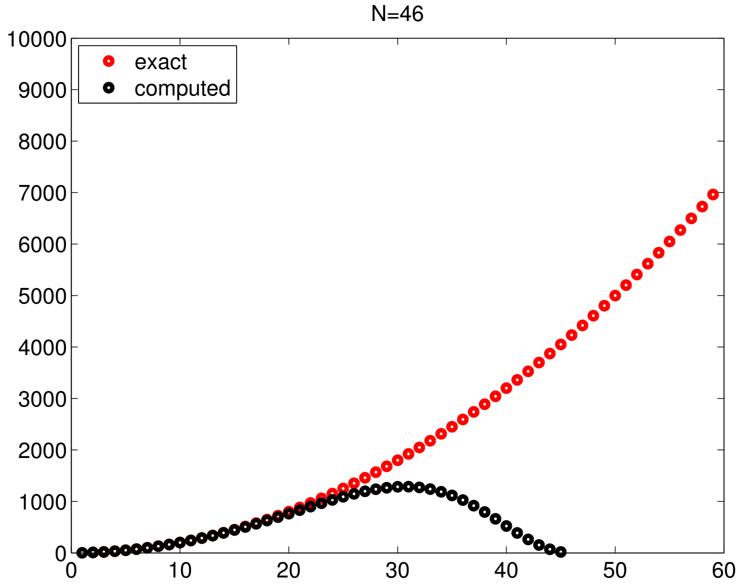

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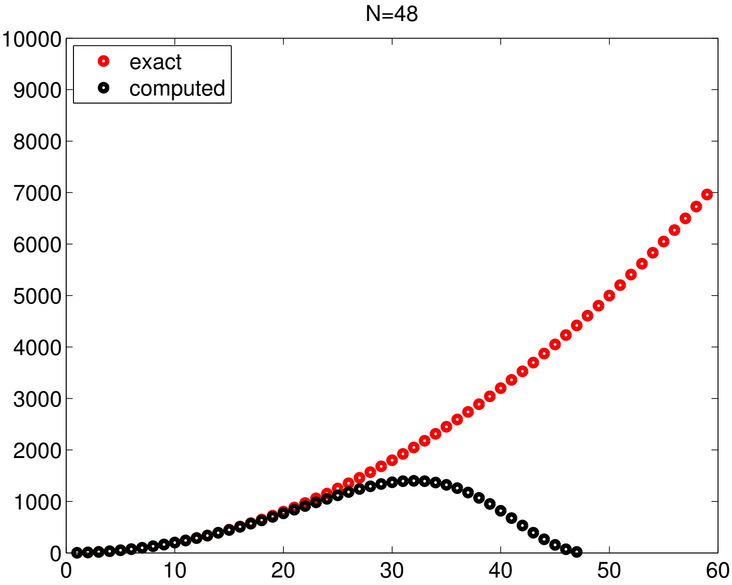

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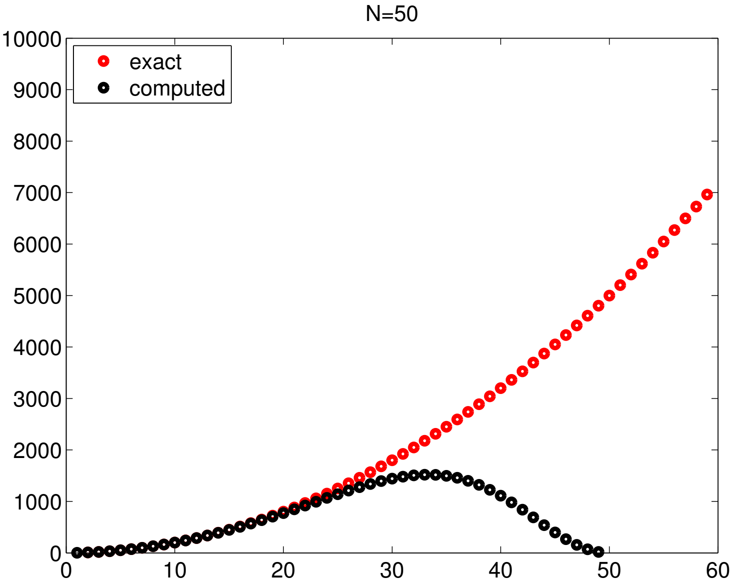

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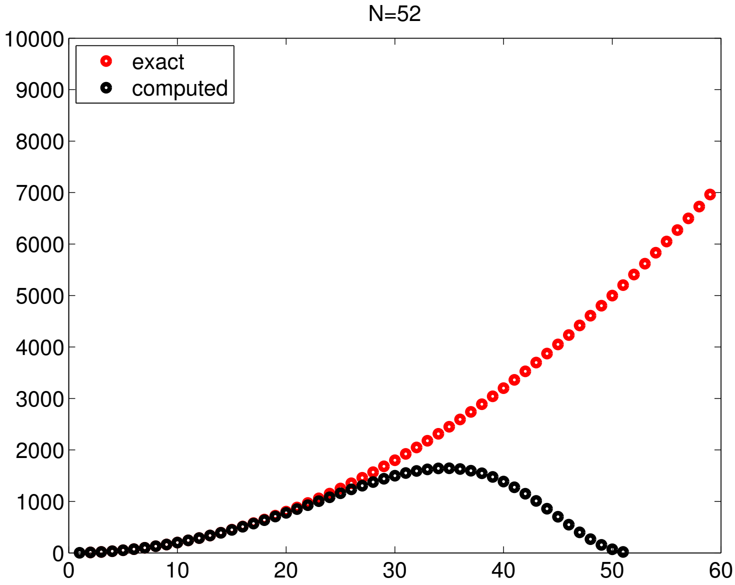

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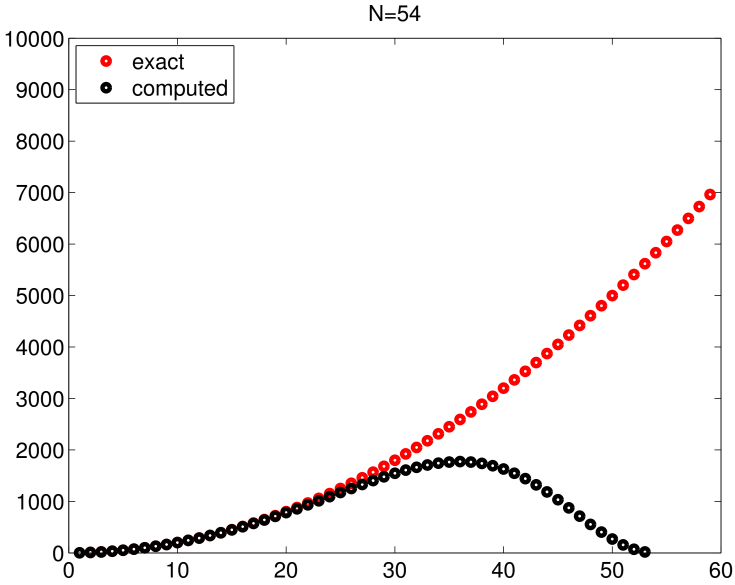

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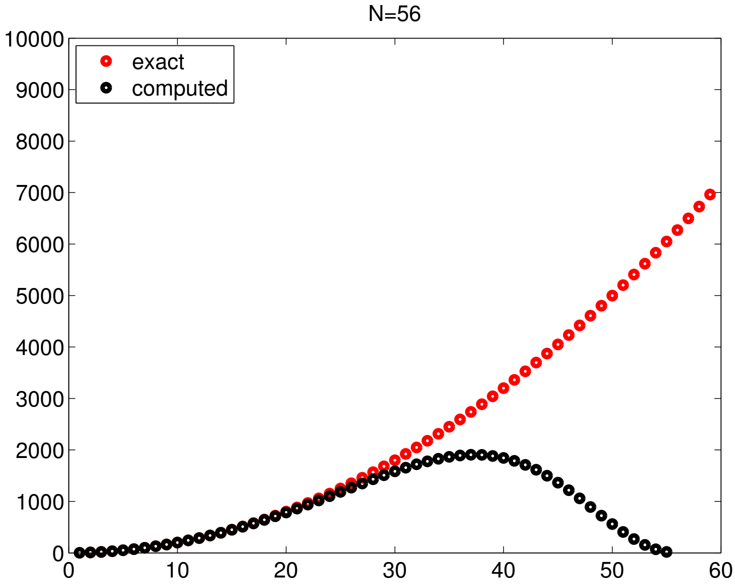

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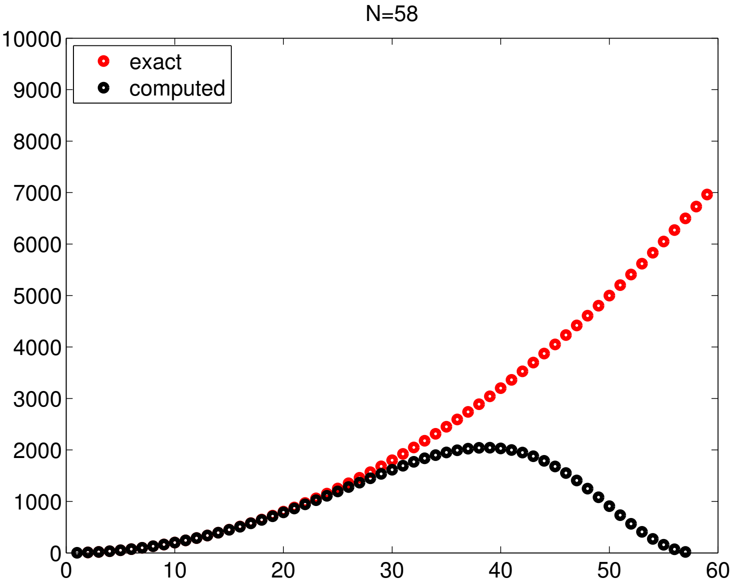

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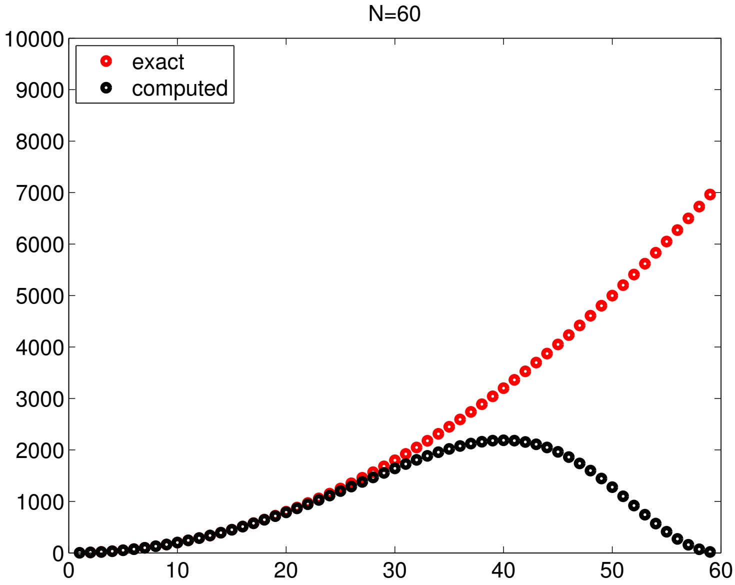


Convergence in norm

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

Theorem

If T is selfadjoint and compact

Uniform convergence $\Longleftrightarrow$ *Eigenmodes convergence*

Strategy

1) prove uniform convergence,
2) estimate the order of convergence

Definition of the solution operator

⟨B.–Brezzi–Gastaldi '97⟩

$T : L^2(\Omega) \to L^2(\Omega)$

$\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)$, $Tg \in L^2(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, Tg) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -(g, v) & \forall v \in L^2(\Omega) \end{cases}$$

Operator is compact; standard mixed estimates don't help

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{\mathrm{div}} + ||u - u_h||_0 \le C \inf_{\boldsymbol{\tau}_h, v_h} (\underbrace{||\boldsymbol{\sigma} - \boldsymbol{\tau}_h||_{\mathrm{div}}}_{O(1)} + \underbrace{||u - v_h||_0}_{O(h)})$$

$$\inf_{\boldsymbol{\tau}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{div}} \le Ch^k(\|\boldsymbol{\sigma}\|_k + \|\,\mathrm{div}\,\boldsymbol{\sigma}\|_k)$$

$$\inf_{\boldsymbol{\tau}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_0 \le Ch^k\|\boldsymbol{\sigma}\|_k$$

Fundamental comment

We need an estimate for u_h which does not involve $\mathrm{div}\,\boldsymbol{\sigma}$

Uniform convergence $\|T - T_h\| \to 0$

The following refined error estimate can be obtained

⟨Falk–Osborn '90⟩

Theorem

$$||u - u_h||_0 \leq C \left(\inf_{v_h \in U_h} ||u - v_h||_0 + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{L^2} \right)$$

Corollary

If the finite element scheme fulfills the commuting diagram property, then the uniform convergence is satisfied

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq \|\boldsymbol{\sigma} - \Pi_\Sigma \boldsymbol{\sigma}\|_0$$

Outline

Main references

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. *Finite element exterior calculus, homological techniques, and applications.* Acta Numer., 15:1–155, 2006

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. *Finite element exterior calculus: from Hodge theory to numerical stability.* Bull. Amer. Math. Soc. (N.S.), 47:281–354, 2010

Algebraic k-forms

Skew-symmetric k-linear form $F : (\mathbb{R}^n)^k \rightarrow \mathbb{R}$
$(v_1, \ldots, v_k) \mapsto F(v_1, \ldots, v_k)$

For instance, for $n = 3$, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$

$dx(\mathbf{u}) = u_1$ is a 1-form
$dx \wedge dy(\mathbf{u}, \mathbf{v}) = u_1 v_2 - u_2 v_1$ is a 2-form

Differential k-forms

Given $\Omega \subset \mathbb{R}^n$, a differential k-form is a field of algebraic k-forms

The de Rham complex

1D

$$0 \to C^\infty(\Omega) \xrightarrow{d/dx} C^\infty(\Omega) \to 0$$

2D

$$0 \to C^\infty(\Omega) \xrightarrow{\mathbf{grad}} C^\infty(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{rot}} C^\infty(\Omega) \to 0$$

3D

$$0 \to C^\infty(\Omega) \xrightarrow{\mathbf{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\mathbf{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} C^\infty(\Omega) \to 0$$

n-D

$$0 \to \Lambda^0(\Omega) \xrightarrow{d^0} \Lambda^1(\Omega) \xrightarrow{d^1} \Lambda^2(\Omega) \cdots\cdots \xrightarrow{d^{n-1}} \Lambda^n(\Omega) \to 0$$

$\Lambda^k(\Omega) = C^\infty(\Omega, (\mathbb{R}^n)^k_{skw})$ is the space of smooth differential k-forms on Ω

$d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ is the exterior derivative

Hilbert complex

A Hilbert complex is a sequence of Hilbert spaces W^k and closed linear operators $d^k : W^k \to W^{k+1}$ such that the range of d^k is contained in the kernel of d^{k+1}

$V_k = D(d^k)$ with graph norm $\|v\|_{V^k}^2 = \|v\|_{W^k}^2 + \|d^k v\|_{W^{k+1}}^2$ gives rise to the domain (cochain) complex

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} V^n \to 0$$

If $d_k^* : V_k^* \subset W^k \to W^{k-1}$ is the adjoint of d^{k-1} then we have the domain (chain) dual complex

$$0 \to V_n^* \xrightarrow{d_n^*} V_{n-1}^* \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_1^*} V_0^* \to 0$$

Standard terminology

Range $\mathcal{B}_k$: (co)boundaries; kernel $\mathcal{C}_k$: (co)cycles
$\mathcal{H}_k = \mathcal{C}_k / \mathcal{B}_k$: (co)homology

Closed range theorem

Theorem

$T : X \to Y$. *If the range of* T *is closed in* Y *then the range of* T^* *is closed in* X

Example: $\Omega \subset \mathbb{R}^3$ smooth so that trace theorem holds

Closed operators and their adjoints

$$
\begin{array}{llll}
\mathbf{grad}: & H^1(\Omega) \to L^2(\Omega) & -\operatorname{div}: & H^1_0(\Omega) \to L^2(\Omega) \\
\mathbf{curl}: & \boldsymbol{H}(\mathbf{curl}) \to L^2(\Omega) & \mathbf{curl}: & \boldsymbol{H}_0(\mathbf{curl}) \to L^2(\Omega) \\
\operatorname{div}: & \boldsymbol{H}(\operatorname{div};\Omega) \to L^2(\Omega) & -\mathbf{grad}: & H^1_0(\Omega) \to L^2(\Omega)
\end{array}
$$

Consequence for Hilbert complexes

(W, d) is closed iff (W, d^*) is closed

[closed means that the (co)boundaries are closed]

Hodge decomposition and Poincaré inequality

Closed Hilbert complex (W, d)

Hodge (orthogonal) decomposition

$$W^k = (\mathcal{B}^k \oplus \mathbf{h}^k) \oplus \mathcal{B}_k^* = \mathcal{Z}^k \oplus \mathcal{Z}^{k\perp}$$

$$V^k = (\mathcal{B}^k \oplus \mathbf{h}^k) \oplus \mathcal{Z}^{k\perp_V}$$

Poincaré inequality

$$\|z\|_V \le C\|dz\|_W \quad \forall z \in \mathcal{Z}^{k\perp_V}$$

Examples:

$\|v\|_{H^1} \le C\|\,\mathbf{grad}\,v\|_{L^2} \quad \forall v \in H^1(\Omega) \cap L_0^2(\Omega)$

$\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl})} \le C\|\mathbf{v}\|_{L^2} \quad \forall \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl}) : (\mathbf{v}, \mathbf{grad}\,\phi) = 0 \quad \forall \phi \in H_0^1$

Abstract Hodge Laplacian

$\boxed{d^*d + dd^* : W^k \to W^k}$ $f \in W^k$ given

Primal formulation

Find $u \in V^k \cap V_k^* \cap \mathbf{h}^{k\perp}$:

$$(du, dv) + (d^*u, d^*v) = (f - P_{\mathbf{h}}f, v) \quad \forall v \in V^k \cap V_k^* \cap \mathbf{h}^{k\perp}$$

Mixed formulation

Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathbf{h}^k$:

$$\begin{cases} (\sigma, \tau) + (d\tau, u) = 0 & \forall \tau \in V^{k-1} \\ (d\sigma, v) - (du, dv) - (p, v) = -(f, v) & \forall v \in V^k \\ (u, q) = 0 & \forall q \in \mathbf{h}^k \end{cases}$$

$[\sigma = -d^*u, p = P_{\mathbf{h}}u]$

$$\|u\|_W + \|du\|_W + \|d^*u\|_W + \|dd^*u\|_W + \|d^*du\|_W \le C\|f - P_{\mathbf{h}}f\|_W$$

$n = 3, k = 0, 1, 2, 3$

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \to 0$$

$$0 \leftarrow L^2 \xleftarrow{-div} \mathring{H}(\text{div}) \xleftarrow{\text{curl}} \mathring{H}(\text{curl}) \xleftarrow{-\text{grad}} \mathring{H}^1 \leftarrow 0$$

k	$L^k = d^*d + dd^*$	BCs imposed on...		$V^{k-1} \times V^k$
0	$-\Delta$		$\partial u / \partial n$	H^1
1	$\text{curl}\,\text{curl} - \text{grad}\,\text{div}$	$u \cdot n$	$\text{curl}\, u \times n$	$H^1 \times H(\text{curl})$
2	$-\text{grad}\,\text{div} + \text{curl}\,\text{curl}$	$u \times n$	$\text{div}\, u$	$H(\text{curl}) \times H(div)$
3	$-\Delta$	u		$H(\text{div}) \times L^2$

essential BC for primal form. ↗ ↖ natural BC for primal form.

(Courtesy of D. Arnold)

Outline

Mixed Laplace

Eigenvalues problems in mixed form

Differential forms and de Rham complex

Discretization of differential forms

Time harmonic Maxwell's equations

Mixed finite elements for linear elasticity

Discrete k-forms

$$\boxed{V_h^k \subset V^k}$$

Approximation property

$$\lim_{h\to 0} \inf_{v \in V_h^k} \|w - v\|_V = 0 \quad \forall w \in V^k$$

Discrete subcomplex property

$$0 \to V_h^0 \xrightarrow{d^0} V_h^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} V_h^n \to 0$$

Coboundaries, cocycles, harmonic forms

$$\mathcal{B}_h^k \subset \mathcal{B}^k, \qquad \mathcal{C}_h^k \subset \mathcal{C}^k, \qquad \boxed{\mathbf{h}_h^k \not\subset \mathbf{h}^k}$$

Approximation of Hodge Laplacian

Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathbf{h}_h^k$:

$$\begin{cases} (\sigma_h, \tau) + (d\tau, u_h) = 0 & \forall \tau \in V_h^{k-1} \\ (d\sigma_h, v) - (du_h, dv) - (p_h, v) = -(f, v) & \forall v \in V_h^k \\ (u_h, q) = 0 & \forall q \in \mathbf{h}_h^k \end{cases}$$

Bounded cochain projection ($\|\pi_h v\|_V \leq C\|v\|_V$)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V^0 & \xrightarrow{d^0} & V^1 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & V^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_h^0} & & \downarrow{\scriptstyle \pi_h^1} & & & & \downarrow{\scriptstyle \pi_h^n} & & \\ 0 & \longrightarrow & V_h^0 & \xrightarrow{d^0} & V_h^1 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & V_h^n & \longrightarrow & 0 \end{array}$$

Consequences of bounded cochain projection

Discrete (orthogonal) Hodge decomposition

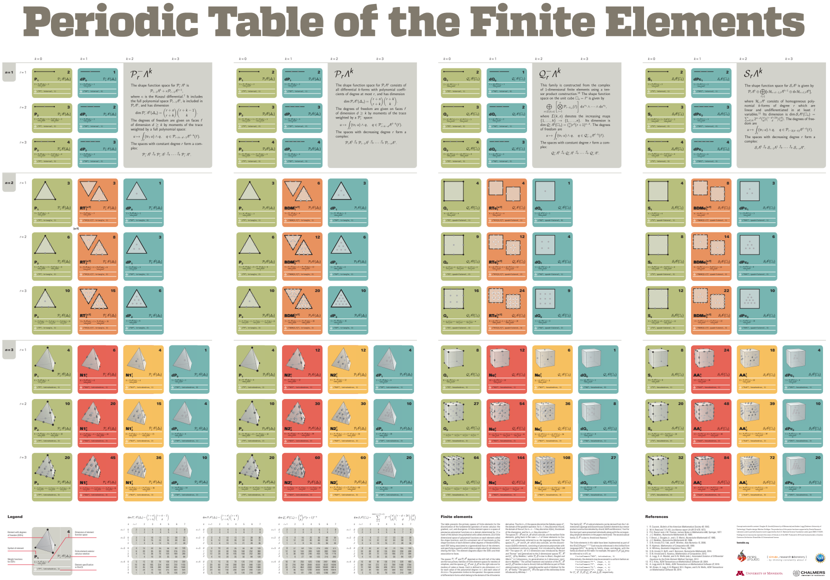
$$V_h^k = \mathcal{B}_h^k \oplus \mathbf{h}_h^k \oplus \mathcal{B}_{k,h}^*$$

Discrete Poincaré inequality

$$\|z\|_V \le C\|dz\|_W \quad \forall z \in \mathcal{Z}_h^{k\perp_V}$$

Quasi-optimal error estimates

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_V \le$$
$$C \inf_{\tau_h, v_h, q_h} (\|\sigma - \tau_h\|_V + \|u - v_h\|_V + \|p - q_h\|_V) +$$
$$C \sup_{r \in \mathbf{h}^k, \|r\|=1} \|(I - \pi_h)r\| \inf_{v_h} \|P_{\mathcal{B}}u - v_h\|_V$$



⟨Arnold–Logg '14⟩

$\mathcal{P}_r^- \Lambda^k$

The shape function space for $\mathcal{P}_r^- \Lambda^k$ is

$$\mathcal{P}_{r-1}\Lambda^k + \kappa \mathcal{P}_{r-1}\Lambda^{k+1},$$

where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

$$\dim \mathcal{P}_r^- \Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^- \Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_r^- \Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r^- \Lambda^n.$$

Periodic table of the finite elements

***n* = 1**

r = 1

2

P₁ — $\mathcal{P}_1^-\Lambda^0(\Delta_1)$

$2 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_0)}_{1} = 2$

("P", interval, 1)

1

dP₀ — $\mathcal{P}_1^-\Lambda^1(\Delta_1)$

$1 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_1)}_{1} = 1$

("DP", interval, 0)

r = 2

3

P₂ — $\mathcal{P}_2^-\Lambda^0(\Delta_1)$

$2 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_0)}_{1} + 1 \times \underbrace{\mathcal{P}_0\Lambda^1(\Delta_1)}_{1} = 3$

("P", interval, 2)

2

dP₁ — $\mathcal{P}_2^-\Lambda^1(\Delta_1)$

$1 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_1)}_{2} = 2$

("DP", interval, 1)

r = 3

4

P₃ — $\mathcal{P}_3^-\Lambda^0(\Delta_1)$

$2 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_0)}_{1} + 1 \times \underbrace{\mathcal{P}_1\Lambda^1(\Delta_1)}_{2} = 4$

("P", interval, 3)

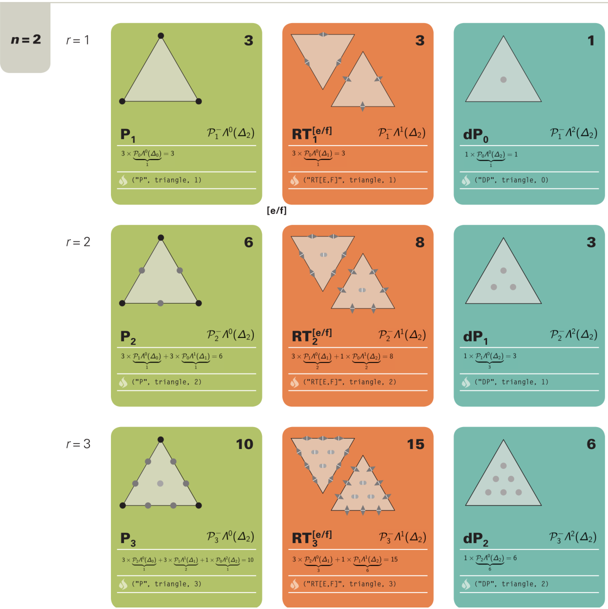
3

dP₂ — $\mathcal{P}_3^-\Lambda^1(\Delta_1)$

$1 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_1)}_{3} = 3$

("DP", interval, 2)

Periodic table of the finite elements

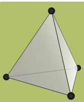


Periodic table of the finite elements

***n* = 3**

$r = 1$

4

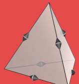


P$_1$ $\mathcal{P}_1^- \Lambda^0(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_0)}_{1} = 4$

("P", tetrahedron, 1)

6

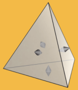


N1$^e_1$ $\mathcal{P}_1^- \Lambda^1(\Delta_3)$

$6 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_1)}_{1} = 6$

("N1E", tetrahedron, 1)

4

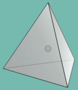


N1$^f_1$ $\mathcal{P}_1^- \Lambda^2(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_2)}_{1} = 4$

("N1F", tetrahedron, 1)

1



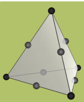
dP$_0$ $\mathcal{P}_1^- \Lambda^3(\Delta_3)$

$1 \times \underbrace{\mathcal{P}_0\Lambda^0(\Delta_3)}_{1} = 1$

("DP", tetrahedron, 0)

$r = 2$

10

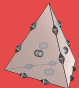


P$_2$ $\mathcal{P}_2^- \Lambda^0(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_0)}_{1} + 6 \times \underbrace{\mathcal{P}_0\Lambda^1(\Delta_1)}_{1} = 10$

("P", tetrahedron, 2)

20

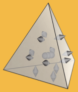


N1$^e_2$ $\mathcal{P}_2^- \Lambda^1(\Delta_3)$

$6 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_1)}_{2} + 4 \times \underbrace{\mathcal{P}_0\Lambda^1(\Delta_2)}_{2} = 20$

("N1E", tetrahedron, 2)

15

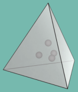


N1$^f_2$ $\mathcal{P}_2^- \Lambda^2(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_2)}_{3} + 1 \times \underbrace{\mathcal{P}_0\Lambda^1(\Delta_3)}_{3} = 15$

("N1F", tetrahedron, 2)

4



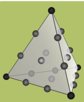
dP$_1$ $\mathcal{P}_2^- \Lambda^3(\Delta_3)$

$1 \times \underbrace{\mathcal{P}_1\Lambda^0(\Delta_3)}_{4} = 4$

("DP", tetrahedron, 1)

$r = 3$

20

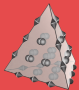


P$_3$ $\mathcal{P}_3^- \Lambda^0(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_0)}_{1} + 6 \times \underbrace{\mathcal{P}_1\Lambda^1(\Delta_1)}_{2} + 4 \times \underbrace{\mathcal{P}_0\Lambda^2(\Delta_2)}_{1} = 20$

("P", tetrahedron, 3)

45

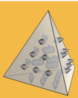


N1$^e_3$ $\mathcal{P}_3^- \Lambda^1(\Delta_3)$

$6 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_1)}_{3} + 4 \times \underbrace{\mathcal{P}_1\Lambda^1(\Delta_2)}_{6} + 1 \times \underbrace{\mathcal{P}_0\Lambda^2(\Delta_3)}_{3} = 45$

("N1E", tetrahedron, 3)

36

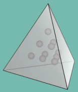


N1$^f_3$ $\mathcal{P}_3^- \Lambda^2(\Delta_3)$

$4 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_2)}_{6} + 1 \times \underbrace{\mathcal{P}_1\Lambda^1(\Delta_3)}_{12} = 36$

("N1F", tetrahedron, 3)

10



dP$_2$ $\mathcal{P}_3^- \Lambda^3(\Delta_3)$

$1 \times \underbrace{\mathcal{P}_2\Lambda^0(\Delta_3)}_{10} = 10$

("DP", tetrahedron, 2)

Back to the $L^2(\Omega)$ estimate

Improved error estimates (in the norm of W instead of V) follow from the W-boundedness of the cochain projections (stronger requirement than the V-boundedness)

$$\|\pi_h v\|_W \le C\|v\|_W$$

Remark

The V-boundedness of the cochain projections is related to the inf-sup condition and to the estimates in the energy norm; the W-boundedness of the cochain projection is related to the commuting diagram property, to the optimal estimates in $L^2(\Omega)$, and to the eigenvalue problems

Some connection with the standard theory

We said:

A contravariant mapping from the reference element (*Piola transform*) is used for their definitions

$$\boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} DF(\hat{\mathbf{x}})\hat{\boldsymbol{\sigma}}(\hat{\mathbf{x}})$$

Moreover by construction the spaces satisfy exactly $\operatorname{div} \Sigma_h = U_h$

Mapping of discrete differential forms:

$$\mathbf{x} : \widehat{K} \to \mathbb{R}^n$$

$$\hat{x} \mapsto \mathbf{x}(\hat{x})$$

$$K = \mathbf{x}(\widehat{K})$$

$\widehat{V}_h^k$ reference space, V_h^k mapped space:

$$\boxed{\widehat{V}_h^k = \mathbf{x}^* V_h^k}$$

Pullbacks

Pullback of 0-forms

$$\hat{v}(\hat{x}) = v\left(\mathbf{x}(\hat{x})\right)$$

Pullbacks of generic constant
1-forms dx^i $(i \in \{1, \ldots, n\})$
2-forms $dx^i \wedge dx^j$ $(i, j \in \{1, \ldots, n\})$
k-forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ $(i_1, \ldots, i_k \in \{1, \ldots, n\})$

$$\mathbf{x}^*(dx^i) = \frac{\partial x_i}{\partial \hat{x}^j} d\hat{x}^j$$

$$\mathbf{x}^*(dx^i \wedge dx^j) = \frac{\partial x_i}{\partial \hat{x}^k} \frac{\partial x_j}{\partial \hat{x}^l} d\hat{x}^k \wedge d\hat{x}^l$$

$$\mathbf{x}^*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{j_1, \ldots, j_k} \frac{\partial x_{i_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial x_{i_k}}{\partial \hat{x}^{j_k}} d\hat{x}^{j_1} \wedge \cdots \wedge d\hat{x}^{j_k}$$

Outline

The Maxwell eigenvalue problem

Ampère and Faraday's laws: find resonance frequencies $\omega \in \mathbb{R}$ (with $\omega \neq 0$) and electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= i\omega\mu\mathbf{H} && \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} &= -i\omega\varepsilon\mathbf{E} && \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} &= 0 && \text{on } \partial\Omega \\ \mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

$\omega \neq 0$ gives divergence conditions

$$\begin{aligned} \operatorname{div} \varepsilon\mathbf{E} &= 0 && \text{in } \Omega \\ \operatorname{div} \mu\mathbf{H} &= 0 && \text{on } \Omega \end{aligned}$$

It is then standard to eliminate one field and to obtain the **curl curl** problem

Eliminate $\mathbf{H}$ and take $\mathbf{u} = \mathbf{E}$, $\lambda = \omega^2$

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in } \Omega \\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Well-known and intensively studied problem. Special (*edge*) finite elements required for its approximation.

Edge elements are discrete 1-forms

For ease of presentation, we take $\mu = \varepsilon = 1$ and simple topology from now on.

Standard formulation

The standard variational formulation reads

$$\begin{aligned} &\lambda \in \mathbb{R},\ \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\ &(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ &(\mathbf{u}, \mathbf{grad}\,\phi) = 0 \qquad \forall \phi \in H_0^1 \end{aligned}$$

The most commonly used variational formulation is based on the replacement of the divergence free constraint by the condition $\lambda \neq 0$

$$\begin{aligned} &\lambda \in \mathbb{R} \setminus 0,\ \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\ &(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \end{aligned}$$

The kernel $\lambda = 0$ corresponds to the infinite dimensional space $\mathbf{grad}\, H_0^1$ (N.B.: Helmholtz decomposition)

⟨Kikuchi '89⟩

Divergence free constraint imposed via Lagrange multiplier ψ

$$\lambda \in \mathbb{R},\ \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}),\ \psi \in H_0^1 :$$

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{v}, \mathbf{grad}\,\psi) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{u}, \mathbf{grad}\,\phi) = 0 & \forall \phi \in H_0^1 \end{cases}$$

⟨B.–Fernandes–Gastaldi–Perugia '99⟩

Second mixed formulation ($\mathbf{H}_0(\mathrm{div}^0) = \mathbf{curl}(\mathbf{H}_0(\mathbf{curl}))$)

$$\lambda \in \mathbb{R},\ \boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{curl}),\ \boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{div}^0) :$$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{\sigma}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \boldsymbol{\tau}) = -\lambda(\boldsymbol{\sigma}, \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div}^0) \end{cases}$$

The “standard” language of FEM

Let’s see how the problem can be analyzed within the framework of “standard” finite elements

Commuting diagram property (de Rham complex)

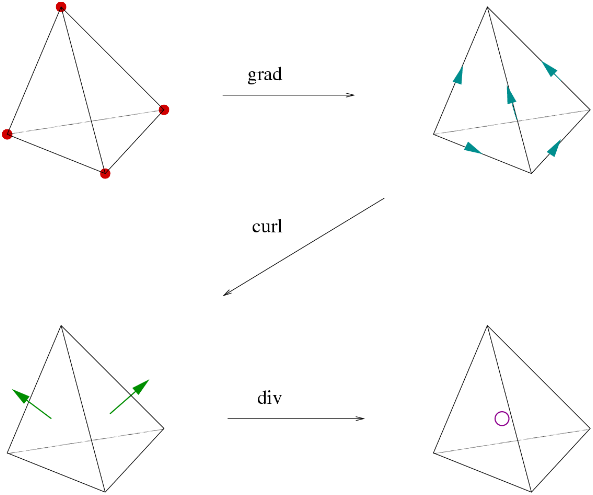
⟨Douglas–Roberts '82⟩
⟨Bossavit '88⟩
⟨Arnold '02⟩

$Q \subset H_0^1,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$
\begin{array}{ccccccccc}
0 \to & Q & \xrightarrow{\mathbf{grad}} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\
& \downarrow \Pi_h^Q & & \downarrow \Pi_h^V & & \downarrow \Pi_h^U & & \downarrow \Pi_h^S & \\
0 \to & Q_h & \xrightarrow{\mathbf{grad}} & V_h & \xrightarrow{\mathbf{curl}} & U_h & \xrightarrow{\mathrm{div}} & S_h & \to 0
\end{array}
$$

- Kikuchi formulation uses Q and V
- Alternative formulation uses V and U
- U and S are used for Darcy flow or mixed Laplacian

Finite elements



See also: FEEC (Finite Element Exterior Calculus)

⟨Arnold–Falk–Winther '10⟩

Kikuchi resolvent operators: continuous...

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{grad}\,q, \mathbf{u}) = 0 & \forall q \in H^1_0 \end{cases}$$

$T^{Ki} \in \mathcal{L}(L^2)$: $T^{Ki}(\mathbf{f}) = \mathbf{u}$

... and discrete one

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,p_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in V_h \\ (\mathbf{grad}\,q, \mathbf{u}_h) = 0 & \forall q \in Q_h \end{cases}$$

$T^{Ki}_h \in \mathcal{L}(L^2)$: $T^{Ki}_h(\mathbf{f}) = \mathbf{u}_h$

$\langle$B.–Brezzi–Gastaldi '97$\rangle$

Theorem

If the ellipticity in the discrete kernel [ELKER], the weak approximability of Q [WA1], and the strong approximability of V_0 [SA1] are satisfied, then the following convergence in norm holds true

$$\|T^{Ki} - T_h^{Ki}\|_{\mathcal{L}(L^2)} \to 0$$

Remark

Convergence in norm allows us to use the classical Babuška–Osborn theory for eigenmode convergence

Alternative resolvent operators: continuous…

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -(\mathbf{g}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{curl}(\mathbf{H}_0(\mathbf{curl})) \end{cases}$$

$T^{M2} \in \mathcal{L}(L^2)$: $T^{M2}(\mathbf{g}) = \mathbf{z}$

…and discrete one

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}_h) = 0 & \forall \boldsymbol{\tau} \in V_h \\ (\mathbf{curl}\,\boldsymbol{\sigma}_h, \mathbf{w}) = -(\mathbf{g}, \mathbf{w}) & \forall \mathbf{w} \in Z_h \end{cases}$$

$T_h^{M2} \in \mathcal{L}(L^2)$: $T_h^{M2}(\mathbf{g}) = \mathbf{z}_h$

⟨B.–Brezzi–Gastaldi '97⟩

Theorem

If the weak approximability of Z^0 *[WA2] and the strong approximability of* Z^0 *[SA2] are satisfied, and if there exists a Fortin operator satisfying the Fortid property [FORTID], then the following convergence in norm holds true*

$$\|T^{M2} - T_h^{M2}\|_{\mathcal{L}(L^2)} \to 0$$

Remark

In the language of FEEC, all these conditions boil down to the existence of W*-bounded cochain projectors.*

Outline

The approximation of linear elasticity is an example where the language of FEEC made it possible to design new elements that would have been difficult to invent using the standard framework

N.B.
Slides on FEEC courtesy of D. Arnold

displacement $u : \Omega \to \mathbb{V} := \mathbb{R}^n$ $\quad A\sigma = \epsilon\, u := [\nabla u + (\nabla u)^T]/2$

stress $\sigma : \Omega \to \mathbb{S} := \mathbb{R}^{n\times n}_{\text{sym}}$ $\quad \operatorname{div}\sigma = f$

$\sigma \in H(\operatorname{div}, \Omega; \mathbb{S}), u \in L^2(\Omega; \mathbb{V})$ satisfy

$$\int_\Omega A\sigma : \tau\, dx + \int_\Omega \operatorname{div}\tau \cdot u\, dx = 0 \qquad \forall \tau \in H(\operatorname{div}, \Omega; \mathbb{S})$$

$$\int_\Omega \operatorname{div}\sigma \cdot v\, dx = \int_\Omega f \cdot v \qquad \forall v \in L^2(\Omega; \mathbb{V})$$

$(\sigma, u) \in H(\operatorname{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{V})$ saddle point of

$$\mathcal{L}(\tau, v) = \int_\Omega \big(\frac{1}{2} A\tau : \tau + \operatorname{div}\tau \cdot v - f \cdot v\big)\, dx.$$

The elasticity complexes

A key to developing stable elements for elasticity (with strongly imposed symmetry) is the *elasticity complex*:

$$\mathbb{T} \hookrightarrow \underset{\substack{\uparrow \\ \text{displacement}}}{C^\infty(\Omega,\mathbb{V})} \xrightarrow{\epsilon} \underset{\substack{\uparrow \\ \text{strain}}}{C^\infty(\Omega,\mathbb{S})} \xrightarrow{J} \underset{\substack{\uparrow \\ \text{stress}}}{C^\infty(\Omega,\mathbb{S})} \xrightarrow{\text{div}} \underset{\substack{\uparrow \\ \text{load}}}{C^\infty(\Omega,\mathbb{V})} \longrightarrow 0$$

$J = \text{curl}_\text{c}\,\text{curl}_\text{r}$, second order

$\mathbb{T}$ is the space of infinitesmal rigid motions

For weakly imposed symmetry the relevant sequence is

$$\mathbb{T} \hookrightarrow C^\infty(\mathbb{V}\times\mathbb{K}) \xrightarrow{(\text{grad},-I)} C^\infty(\mathbb{M}) \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{\begin{pmatrix}\text{div}\\ \text{skw}\end{pmatrix}} C^\infty(\mathbb{V}\times\mathbb{K}) \longrightarrow 0$$

where J is defined to be zero on skew matrices.

Bernstein–Gelfand–Gelfand construction, I

1. Start with the de Rham sequence with values in $\mathbb{W} := \mathbb{K} \times \mathbb{V}$:

$$\mathbb{W} \hookrightarrow \Lambda^0(\Omega;\mathbb{W}) \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \Lambda^1(\Omega;\mathbb{W}) \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \Lambda^2(\Omega;\mathbb{W}) \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \Lambda^3(\Omega;\mathbb{W}) \longrightarrow 0$$

2. For any $x \in \mathbb{R}^3$ define $K_x : \mathbb{V} \to \mathbb{K}$ by $K_x v = 2\,\mathrm{skw}(xv^T)$ and $K : \Lambda^k(\Omega;\mathbb{V}) \to \Lambda^k(\Omega;\mathbb{K})$ by

$$(K\omega)_x(v_1,\dots,v_k) = K_x[\omega_x(v_1,\dots,v_k)].$$

3. Define automorphisms $\Phi : \Lambda^k(\mathbb{W}) \to \Lambda^k(\mathbb{W})$ by

$$\Phi = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}$$

4. Define $\mathcal{A} = \Phi \circ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \circ \Phi^{-1}$ to get a modified de Rham sequence:

$$\Phi(\mathbb{W}) \hookrightarrow \Lambda^0(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^1(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^2(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^3(\mathbb{W}) \to 0$$

Bernstein–Gelfand–Gelfand construction, II

5. Note that $\mathcal{A} = \begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}$, where $S = dK - Kd : \Lambda^k(\Omega; \mathbb{V}) \to \Lambda^{k+1}(\Omega; \mathbb{K})$ is given by

$$(S\omega)_x(v_1, \dots, v_{k+1}) = \sum_\mu \operatorname{sign}(\mu) K_{v_{\mu_{k+1}}} \omega_x(v_{\mu_1}, \dots, v_{\mu_k}).$$

Properties: S is algebraic; for $k = 1$, S is an isomorphism; $dS = -Sd$
$d(dK - Kd) = -dKd = -(dK - Kd)d$

6. Define subspaces $\Gamma^k \subset \Lambda^k(\Omega; \mathbb{W})$ satisfying $\mathcal{A}(\Gamma^k) \subset \Gamma^{k+1}$ and projections

$$\pi_k : \Lambda^k(\Omega; \mathbb{W}) \to \Gamma^k \qquad \text{satisfying} \qquad \pi_{k+1}\mathcal{A} = \mathcal{A}\pi_k :$$

$$\Gamma^0 = \Lambda^0(\Omega; \mathbb{W}), \quad \pi_0 = id, \quad \Gamma^3 = \Lambda^3(\Omega; \mathbb{W}), \quad \pi_3 = id,$$

$$\Gamma^1 = \{\, (\omega, \mu) \in \Lambda^1(\Omega; \mathbb{W}) : d\omega = S\mu \,\}, \quad \Gamma^2 = \{\, (\omega, \mu) \in \Lambda^2(\Omega; \mathbb{W}) : \omega = 0 \,\}$$

$$\pi^1 = \begin{pmatrix} I & 0 \\ S^{-1}d & 0 \end{pmatrix} : \Lambda^1(\Omega; \mathbb{W}) \to \Gamma^1, \quad \pi^2 = \begin{pmatrix} 0 & 0 \\ dS^{-1} & I \end{pmatrix} : \Lambda^2(\Omega; \mathbb{W}) \to \Gamma^2.$$

Bernstein–Gelfand–Gelfand construction, III

6. The following diagram commutes (use $dS = -Sd$):

$$
\begin{array}{cccccccccc}
\Phi(\mathbb{W}) \hookrightarrow & \Lambda^0(\mathbb{W}) & \xrightarrow{\mathcal{A}} & \Lambda^1(\mathbb{W}) & \xrightarrow{\mathcal{A}} & \Lambda^2(\mathbb{W}) & \xrightarrow{\mathcal{A}} & \Lambda^3(\mathbb{W}) & \to & 0 \\
& \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi_3} & & \\
\Phi(\mathbb{W}) \hookrightarrow & \Gamma^0 & \xrightarrow{\mathcal{A}} & \Gamma^1 & \xrightarrow{\mathcal{A}} & \Gamma^2 & \xrightarrow{\mathcal{A}} & \Gamma^3 & \to & 0
\end{array}
$$

Therefore, the subcomplex on the bottom row is exact.

7. This subcomplex may be identified with the elasticity complex.

Bernstein–Gelfand–Gelfand construction, concluded

$$\begin{array}{ccccccc}
\Gamma^0 & \xrightarrow{\mathcal{A}} & \Gamma^1 & \xrightarrow{\mathcal{A}} & \Gamma^2 & \xrightarrow{\mathcal{A}} & \Gamma^3 \\
= & & \cong & & \cong & & = \\
\Lambda^0(\mathbb{K}\times\mathbb{V}) & \xrightarrow{(d_0,-S_0)} & \Lambda^1(\Omega;\mathbb{K}) & \xrightarrow{d_1\circ S_1^{-1}\circ d_1} & \Lambda^2(\Omega;\mathbb{V}) & \xrightarrow{(-S_2,d_2)^T} & \Lambda^3(\mathbb{K}\times\mathbb{V})
\end{array}$$

With the identifications

$$\Lambda^0(\mathbb{K}\times\mathbb{V}) \leftrightarrow C^\infty(\mathbb{V}\times\mathbb{K})$$
$$\Lambda^1(\mathbb{K}) \leftrightarrow C^\infty(\mathbb{M})$$
$$\Lambda^2(\mathbb{K}) \leftrightarrow C^\infty(M)$$
$$\Lambda^3(\mathbb{K}\times\mathbb{V}) \leftrightarrow C^\infty(\mathbb{V}\times\mathbb{K})$$

this becomes the elasticity sequence

$$\mathcal{T}' \hookrightarrow C^\infty(\mathbb{V}\times\mathbb{K}) \xrightarrow{(\mathrm{grad},-I)} C^\infty(\mathbb{M}) \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{(\mathrm{div},\mathrm{skw})^T} C^\infty(\mathbb{V}\times\mathbb{K}) \to 0$$

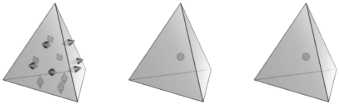
Stable elements

The requirement that $\bar{\pi}_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to \bar{V}_h^1 \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}$ is surjective can be checked by looking at DOFs.

The simplest choice is

$$\mathcal{P}_r^- \Lambda^{n-1} \xrightarrow{\mathrm{div}} \mathcal{P}_r^- \Lambda^n \to 0, \quad \mathcal{P}_{r+1}^- \Lambda^{n-2} \xrightarrow{\mathrm{curl}} \mathcal{P}_r \Lambda^{n-1} \xrightarrow{-\mathrm{div}} \mathcal{P}_{r-1} \Lambda^n \to 0$$

This gives the elements of DNA–Falk–Winther '07



σ u p

Other elements:
Cockburn–Gopalakrishnan–Guzmán,
Gopalakrishnan–Guzmán, Stenberg, ...

Standard analysis

⟨B–Brezzi–Fortin '09⟩

The elements by Arnold Falk and Winther can be analyzed by standard techniques but it would have been difficult to figure out how to construct them without the Bernstein–Gelfand–Gelfand complex

$$(BDM_k)^3 - (P_{k-1})^3 - S^3(P_{k-1})^3$$

Conclusions

The FEEC is a mathematical framework that uses the calculus of differential forms for the formulation of the finite element method

- Rigorous and elegant mathematical setting
- "Standard" finite element analysis of problems in mixed form fits this framework very well
- FEEC allows the design of new finite element schemes