4. AKSZ construction / PTVV construction / transgression

Mapping stack

X, Y stacks. $MAP(X,Y)(\mathrm{Spec}\,A) \overset{\mathrm{def}}{=} \mathrm{Hom}_{dSt}(X \times \mathrm{Spec}\,A, Y)$ defines a stack $MAP(X,Y)$.

Let $x : \mathrm{Spec}\,A \to MAP(X,Y) \Leftrightarrow f_x : X \times \mathrm{Spec}\,A \to Y$

Lemma: $x^* \mathbb{T}_{MAP(X,Y)} \simeq \Gamma(X \times \mathrm{Spec}\,A, f_x^* \mathbb{T}_Y)$.

Assume: ① Y has a ND n-shifted 2-form $\omega_0 : \Lambda^2 \mathbb{T}_Y \to \mathcal{O}_Y[n]$

② X has a d-orientation duality: ⓐ $[X] : \Gamma(X, \mathcal{O}_X) \to k[-d]$

ⓑ E perfect on $X \times \mathrm{Spec}\,A$

$\Rightarrow \Gamma(X \times \mathrm{Spec}\,A, E)$ perfect A-module.

ⓒ E perfect $\Rightarrow \Gamma(X, E^*) \to \Gamma(X, E)^*[-d]$

$\xi \longmapsto (s \mapsto [X](s(\xi)))$

is a quasi-iso.

Then there is a ND $(n-d)$-shifted 2-form $\int_{[X]} \omega_0$ on $MAP(X,Y)$:

$\forall x : \mathrm{Spec}\,A \to MAP(X,Y)$, given by $f_x : X \times \mathrm{Spec}\,A \to Y$

$$\Lambda^2_A\, \Gamma(X \times \mathrm{Spec}\,A, f_x^* \mathbb{T}_Y) \xrightarrow{f_x^* \omega_0} \Gamma(X \times \mathrm{Spec}\,A, \mathcal{O}_{X \times \mathrm{Spec}\,A})[n] \simeq \Gamma(X, \mathcal{O}_X) \otimes A[n] \xrightarrow{[X] \otimes \mathrm{id}_A} A[n-d]$$

Thm [Pantev-Toën-Vaquié-Vezzosi]: this also works for closed 2-forms.

Betti stack

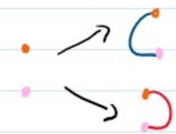
Let X be an ∞-groupoid (for instance any CW complex)

X_B is the sheaf associated with the constant presheaf X.

Remarks: • if X is contractible then $X_B \simeq *$.

- if X is a CW complex, it is a glueing ($\simeq$ colimit) of its cells (that are all $\simeq$ to $*$).

  Since $X \mapsto X_B$ is colimit preserving, $X_B =$ colimit of diagram of points.

  E.g. $S^1_B \cong \mathrm{colim}\Big($  $\Big) = \mathrm{colim}\left(\begin{tikzcd}[row sep=small] * \arrow[r] & * \\ * \arrow[r] & * \end{tikzcd}\right)$

  Then $\Gamma(X, \mathcal{O}_X) = \lim\big(k\langle \text{diagram of cells wrt inclusion}\rangle\big) = C^\bullet_{\mathrm{cell}}(X, k)$.

  E.g. $\Gamma(X, S^1_B) \cong \lim\left(\begin{tikzcd}[row sep=small] k \arrow[dr] & \\ & k \oplus k \\ k \arrow[ur] & \end{tikzcd}\right) \simeq \mathrm{hofib}\left(\begin{array}{c} k \oplus k \to k \oplus k \\ (x,y) \mapsto (x-y,\, y-x) \end{array}\right)$

X_B has a d-orientation duality iff X is a Poincaré duality space of dim d.

E.g. S^1_B has a 1-orientation duality.

Σ_B has a 2-orientation duality if Σ is an oriented surface.

$\boxed{\Rightarrow}$ if G is an affine alg group with invariant metric then BG is 2-shifted symplectic and

$$\mathrm{MAP}(\underset{\substack{\wr\wr \\ B\mathbb{Z}}}{S^1_B}, BG) = [G/G] \text{ is 1-shifted symplectic.}$$

[Fun fact: $S^1 = B\left(\begin{tikzcd}[row sep=small, column sep=small] & a \arrow[dl, shift left] \arrow[dr, shift left] & \\ b \arrow[rr, "\sim"] & & c \end{tikzcd}\right)$ hence $\mathrm{MAP}(S^1_B, BG) \simeq [G^{\{\swarrow, \to, \nwarrow\}} / G^{\{a,b,c\}}]$]

$\mathrm{MAP}(\Sigma_B, BG) =$ derived stack of G-local systems on Σ is 0-shifted symplectic

$$\simeq \mathrm{MAP}\Big(\mathring{\Sigma}_B \coprod_{S^1_B} *, BG\Big) \simeq \underset{\substack{\uparrow \\ BF_{2g}}}{\mathrm{MAP}(\mathring{\Sigma}_B, BG)} \times_{[G/G]} [*/G]$$

$$\simeq [G^{2g}/G] \times_{[G/G]} [*/G]$$ $\leftarrow$ do you recognize something ???

Relative version

There is a relative version of the above, essentially providing the following:

Thm [C]: there is a symmetric monoidal functor $\mathrm{MAP}((-)_B, X)$ for every n-shifted symplectic stack X:

going from Cob^{or}_d: objects are closed oriented $(d-1)$-dim manifolds.

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hom space are classifying oriented cobordisms of those.

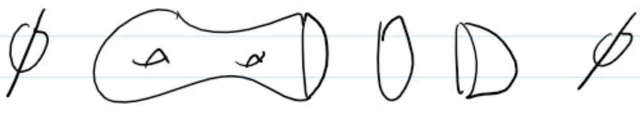
$\otimes = \sqcup$.

to Lag_{n-d+1}: objects are $(n-d+1)$-shifted symplectic stacks

hom spaces are classifying Lagrangian correspondences.

$\otimes = \times$.

For $X = BG$, $n=2$, $d=2$:

 $\longmapsto$

$\mathrm{MAP}(\Sigma_B, BG) \simeq G^{2g} /\!/_1 G$

$$\begin{tikzcd}[column sep=small] & {[G^{2g}/G]} \arrow[dl] \arrow[dr, "\text{Lagrangian}"] & & {[*/G]} \arrow[dl] \arrow[dr] & \\ * & & {[G/G]} & & * \end{tikzcd}$$

Lagrangian (telling us that $G^{2g} \to G$ is a q-Hamiltonian G-space)