3. Shifted symplectic 1-stacks

Higher stacks are sheaves with values in ∞-groupoids (therefore glueing is only up to homotopy).

Q: Sheaves on what? **A**: in diff. geometry: the Euclidean site, with top given by local diffeo.
in alg. geometry: the étale site (affine schemes and étale morphisms).
in derived alg. geometry: the derived étale site (derived affine schemes and étale morphisms).

These are too general gadgets that are not always geometrically tractable.
One restricts to **geometric** stacks: they can be obtained from representable objects (or building blocks) by iterated **smooth groupoid** quotients.

|| def

Simplicial object $X_\bullet$ such that

- $X_1 \rightrightarrows X_0$ are smooth morphisms ($\mathrm{hofib}(\mathbb{T}_{X_1} \to \mathbb{T}_{X_0})$ perfect and concentrated in degree 0)
- $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ equivalence
- $X_2 \to X_1 \times_{X_0} X_1$ are equivalences

If X_n is a manifold (or a smooth affine alg variety) for every n, then this is the same as a Lie groupoid... we will only deal with those, i.e. with underived smooth 1-stacks

De Rham complex

Yoneda lemma tells us that any (pre)sheaf is the colimits of representables mapping to it.

If $\mathcal{X}$ is a stack then $\mathcal{X} = \operatorname{colim}_{\operatorname{Spec} A \to \mathcal{X}} (\operatorname{Spec} A)$.

The quotient of a groupoid $X_\bullet$ is by definition $|X_\bullet| = \operatorname{colim}_{[n] \in \Delta^{op}} (X_n)$

Notation: we also write $|X_\bullet| = [X_0 / X_1]$.

$\llcorner$ whenever $X_0 = *$ then $G = X_1$ is a group and we write $BG = [* / G]$.

REMARK
For honest Lie groupoids
$|X_\bullet| \simeq |Y_\bullet| \Leftrightarrow X_1 \rightrightarrows X_0$ & $Y_1 \rightrightarrows Y_0$ are Morita equivalent

One defines $DR(\mathcal{X}) = \lim_{\operatorname{Spec} A \to \mathcal{X}} (DR(A))$ in the category of graded mixed complexes.
It has an internal differential δ and a de Rham differential d.

Very convenient for theoretical purposes BUT not very practical for computations.

When $\mathcal{X}$ is nice enough (nice $\Rightarrow$ it has a (co)tangent complex), then $DR(\mathcal{X})^{\not{d}} \simeq \Gamma(\mathcal{X}, \mathrm{Sym}_{\mathcal{O}_\mathcal{X}}(\mathbb{L}_\mathcal{X}[-1]))$.

The definitions of (closed) 2-forms and symplectic structures apply verbatim here.
$\Leftarrow$ from the previous lecture.

Concrete description for geometric stacks

Assume that $\mathcal{X} = |X_\bullet|$ for $X_\bullet$ a smooth groupoid (also works for smooth n-groupoids).

Then $DR(\mathcal{X}) = \lim\limits_{[n]\in\Delta} DR(X_n) = DR(X_0) \xrightarrow{\check{\partial}} DR(X_1)[-1] \xrightarrow{\check{\partial}} DR(X)[-2] \to \cdots$

The internal differential is now $\delta + \check{\partial}$ and the de Rham differential is still $d = d_{dR}$.

Let's see what we get for underived smooth 1-stacks: X_n is underived, not stacky and smooth.

There is only $\check{\partial}$ and d $(\delta = 0)$.

($\Rightarrow \mathbb{T}_{X_n} \simeq T_{X_n}$ is concentrated in degree 0).

$\mathbb{L}_{X_n} \simeq \Omega^1_{X_n}$

$$\begin{tikzcd}
\mathcal{O}(X_0) \arrow[d] \arrow[dr] & & \\
\mathcal{O}(X_1) \arrow[d] \arrow[dr] & \Omega^1(X_0) \arrow[d] \arrow[dr] & \\
\mathcal{O}(X_2) \arrow[d] \arrow[dr] & \Omega^1(X_1) \arrow[d] \arrow[dr] & \Omega^2(X_0) \arrow[d] \\
\mathcal{O}(X_3) \arrow[d] \arrow[dr] & \Omega^1(X_2) \arrow[d] \arrow[dr] & \Omega^2(X_1) \arrow[d] \\
\vdots & \Omega^1(X_3) \arrow[d] \arrow[dr] & \Omega^2(X_2) \arrow[d] \\
 & &
\end{tikzcd}$$

weight 2 part

$\downarrow = \check{\partial}$

$\searrow = d$

- A 2-form of degree n in $\mathcal{X} = |X_\bullet|$ is a 2-form $\omega \in \Omega^2(X_n)$ such that $\check{\partial}\omega = 0$.
- A closed 2-form of degree n in $\mathcal{X} = |X_\bullet|$ is a sum $\omega = \omega_0 + \omega_1 + \omega_2 + \cdots + \omega_n$ with $\omega_i \in \Omega^{2+i}(X_{n-i})$ & $(\check{\partial} + d)(\omega) = 0$

How to write concretely the ND condition?

Observe that we have a map $X_0 \xrightarrow{p} |X_\bullet| = \mathcal{X}$

One can check that p^* is conservative (meaning that $E \to F$ quasi-iso $\Leftrightarrow$ $p^*E \to p^*F$ quasi-iso).

Example: $* \xrightarrow{p} BG = [*/G]$ Sheaves on BG are G-reps, and p^* = forgetful map

A G-equiv. map $E \to F$ is a quasi-iso iff it is a quasi-iso of complexes (forgetting the G-action).

Hence ND condition can be checked after taking p^*.

Several observations: ① weight 1 part of $DR(\mathcal{X})$ gives $\Gamma(\mathcal{X}, \mathbb{L}_{\mathcal{X}}[-1])$.

Problem: $\mathcal{X}$ is not a manifold / an affine alg. var. hence this does not determine $\mathbb{L}_{\mathcal{X}}[-1]$.

Solution: apply p^* and land in $\Gamma(X_0, p^*\mathbb{L}_{\mathcal{X}}[-1])$ that determines $p^*\mathbb{L}_{\mathcal{X}}[-1]$.

② $p^*\mathbb{L}_{X_\bullet}$ is a cosimplicial sheaf on X_0, and

$$p^*\mathbb{L}_{\mathcal{X}} = \lim_{[n]\in\Delta}\left(\Omega^1_{X_n}|_{X_0}\right) = \underset{\text{deg } 0}{\Omega^1_{X_0}} \xrightarrow{\check{\partial}} \underset{\text{deg } 1}{\Omega^1_{X_1}|_{X_0}} \to \underset{\text{deg } 2}{\Omega^1_{X_2}|_{X_0}} \to \cdots$$

Dually: $p^*\mathbb{T}_{\mathcal{X}} = \operatorname{colim}_{[n]\in\Delta^{op}}\left(T_{X_n}|_{X_0}\right) = \cdots \to T_{X_1}|_{X_0} \to T_{X_0}$

deg 0 deg 1 deg 2

Dually: $p^*\mathbb{T}_\mathcal{X} = \operatorname{colim}_{[n]\in\Delta^{op}} \left(T_{X_n}|_{X_0}\right) = \cdots \to \underset{\text{deg } -1}{T_{X_1}|_{X_0}} \to \underset{\text{deg } 0}{T_{X_0}}$

(3) Since $X_\bullet$ is a groupoid, $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$ and thus

$$T_{X_n}|_{X_0} = T_{X_1}|_{X_0} \times_{T_{X_0}} \cdots \times_{T_{X_0}} T_{X_1}|_{X_0} \quad \text{and} \quad \Omega^1_{X_n}|_{X_0} = \Omega^1_{X_1}|_{X_0} \oplus_{\Omega^1_{X_0}} \cdots \oplus_{\Omega^1_{X_0}} \Omega^1_{X_1}|_{X_0}$$

$$\left(\cdots \overset{\pi_1}{\underset{\pi_2}{\overset{+}{\rightrightarrows}}} T_{X_1}|_{X_0} \overset{s_*}{\underset{t_*}{\rightrightarrows}} T_{X_0} \right) \qquad \left(\Omega^1_{X_0} \overset{s^*}{\underset{t^*}{\rightrightarrows}} \Omega^1_{X_1}|_{X_0} \overset{(0,id)}{\underset{(id,0)}{\overset{(id,id)}{\rightrightarrows}}} \cdots \right)$$

$$\Rightarrow \quad p^*\mathbb{T}_\mathcal{X} \simeq \underset{(-1)}{T_{X_1}|_{X_0}} \overset{s_*}{\underset{t_*}{\rightrightarrows}} \underset{(0)}{T_{X_0}}$$

$$\underset{\text{q-iso}}{\simeq} \ker(s_*) \xrightarrow{t_* = \text{a anchor map}} T_{X_0}$$

(because s_* surjective) $\quad \overset{\shortparallel}{L}$ Lie algebroid of the groupoid.

$$\Rightarrow \mathrm{Sym}^2\left(p^*\mathbb{L}_\mathcal{X}[-1]\right) \simeq \mathrm{Sym}^2\left(\underset{\text{degree 1}}{\Omega^1_{X_0}} \xrightarrow{a^*} \underset{\text{degree 2}}{L^*}\right)$$

Consequences: there are 4 possibilities

(a) the anchor is an iso, then $p^*\mathbb{L}_\mathcal{X} = 0$, $\mathbb{L}_\mathcal{X} = 0$, and there is only the the 0 form ω, that is symplectic (for any shift).

(b) the anchor is surjective, then $p^*\mathbb{L}_\mathcal{X}$ sits in degree 1 and we can only have a degree 2 form. Let $\mathfrak{g} = \ker(a)$: it is a bundle of Lie algebras.

One can show that

$$\begin{array}{ccc} \Gamma(X_0, \mathrm{Sym}^2(\mathfrak{g}^*[-2]))^{G_1} & \longrightarrow & \left(\widehat{DR}^2/\mathcal{X}, \check{\partial} + d\right) = \text{cpx of closed 2-forms} \\ \downarrow & & \downarrow \; \omega' \mapsto p^*\omega_0 \\ \Gamma(X_0, \mathrm{Sym}^2(\mathfrak{g}^*[-2])) & \xrightarrow{\sim} & \Gamma_0(X_0, \mathrm{Sym}^2(p^*\mathbb{L}_\mathcal{X}[-2])) \end{array}$$

$\langle , \rangle \mapsto \frac{1}{2}\langle p_1^*\theta^L, p_2^*\theta^R \rangle \in \Omega^2(X_1 \times_{X_0} X_1)$

example:
$[*/G] = BG$
any inv. metric on $\mathfrak{g}$ defines a 2-shifted symplectic structure on BG.

(c) the anchor is injective with constant rank, then $p^*\mathbb{L}_\mathcal{X}$ sits in degree 0 and we can only have a degree 0 form $\omega \in \Omega^2(X_0)$ satisfying $d\omega = 0$ and $\check{\partial}\omega = 0$: i.e. ω is constant on the leaves of the regular foliation and transversally ND.

(d) none of the above: then ω must have degree 1, and we recover Ping Xu's notion of a quasi-symplectic groupoid: $\omega = \underbrace{\omega_0}_{\in\,\Omega^2(G_1)} + \underbrace{\omega_1}_{\in\,\Omega^3(G_0)}$.

Lagrangian structures

The definition of Lagrangian structures is the same as in the affine case.

Let $f: X \to Y$ a map of (maybe derived) stacks.

Let ω be an n-shifted symplectic structure on Y.

A **Lagrangian structure** on f is a homotopy η between $f^*\omega$ and 0 such that η_0 (that defines a linear isotropic structure for $f_*: \mathbb{T}_X \to f^*\mathbb{T}_Y$ w.r.t. $f^*\omega$) is ND.

Example [moment maps are Lagrangian morphisms]: Let $X_\bullet$ be a (honest Lie) groupoid and let $Y_\bullet \xrightarrow{f_\bullet} X_\bullet$ (same assumption) an action of $X_\bullet$ on Y_0 (i.e. we assume that $Y_n \xrightarrow{\sim} Y_0 \underset{X_0}{\times} X_n$).

Consequence: if L is the Lie algebroid of $X_\bullet$ (then recall that, on X_0, $p^*\mathbb{T}_{|X_\bullet|} \simeq \underset{-1}{L} \xrightarrow{a} \underset{0}{T_{X_0}}$).

Then $f_0^* L$ is the Lie algebroid of $Y_\bullet$.

We are in the "
$$\begin{tikzcd} E \arrow[r, equal] \arrow[d] & E \arrow[d] \\ B \arrow[r] & F \end{tikzcd}$$
" situation on tangent complexes!

Assume that we have a quasi-symplectic groupoid structure (ω_0, ω_1) on $X_\bullet$

$\Rightarrow$ $\omega_0 + \omega_1$ defines a 1-shifted sympl. structure on $|X_\bullet|$. $\omega_0 \in \Omega^2(X_1)$, $\omega_1 \in \Omega^3(X_0)$

Q: What is a Lagrangian structure on $|f_\bullet|: |X_\bullet| \to |Y_\bullet|$?

A: isotropic means that $\exists\, \eta = \eta_0 \in \Omega^2(X_0)$ s.t. $(\check{\partial} + d)(\eta_0) = f_0^*\omega_0 + f_1^*\omega_1$

(i.e. $\check{\partial}(\eta_0) = f_0^*\omega_0$ & $d\eta_0 = f_1^*\omega_1$)

$\Leftrightarrow$ Y_0 is a pre-Hamiltonian X_1-space in the sense of [P. Xu].

We have already mentioned that the ND condition coincides with Xu's condition

$\Longleftrightarrow$ Y_0 is a pre-Hamiltonian X_1-space in the sense of [P. Xu].

We have already mentionned that the ND condition coincides with Xu's condition for ~~pre~~-Hamiltonian X_1-spaces.

The Yoga of Lagrangian correspondences (in particular Lagrangian intersections) remains the same.

Let's play with it !!!

Examples: • $T^*G \simeq \mathfrak{g}^* \times G \rightrightarrows \mathfrak{g}^*$ coadjoint group. It is a symplectic groupoid. $\Rightarrow$ $[\mathfrak{g}^*/G]$ is 1-shifted sympl.

FACT: Hamiltonian G-spaces are exactly Hamiltonian T^*G-spaces.

Let $M \to \mathfrak{g}^*$ a Hamiltonian G-space. Then $[M/G] \to [\mathfrak{g}^*/G]$ is lagrangian.

Any coadjoint orbit $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a Hamiltonian G-space. $\boxed{\Rightarrow}$:

0-shifted symplectic

$$\begin{tikzcd} M/\!/_{\mathcal{O}}G = [\mathbb{R}\mu^{-1}(\mathcal{O})/G] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {[\mathcal{O}/G]} \arrow[d] \\ {[M/G]} \arrow[r] & {[\mathfrak{g}^*/G]} \end{tikzcd}$$

• if $X_\bullet$ is a symplectic groupoid then the (trivial) isotropic structure on $X_0 \to |X_\bullet|$ is ND. (This actually characterizes symplectic groupoids).

In the above example, this means that $\mathfrak{g}^* \to [\mathfrak{g}^*/G]$ is Lagrangian.

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{g}^* \arrow[d] \\ {[M/G]} \arrow[r] & {[\mathfrak{g}^*/G]} \end{tikzcd}$$

recovers the 0-shifted symplectic structure on M.

• G Lie group with invariant metric on $\mathfrak{g}$.

$G \times G \rightrightarrows G$ adjoint groupoid is quasi-symplectic. $\Rightarrow$ $[G/G]$ 1-shifted symplectic.

FACT: quasi-Hamiltonian G-spaces are exactly Hamiltonian $\left(G\times G \rightrightarrows G\right)$-spaces.

Let $M \to G$ a q-Hamiltonian G-space. Then $[M/G] \to [G/G]$ is lagrangian.

Let $M \to G$ a q-Hamiltonian G-space. Then $[M/G] \to [G/G]$ is lagrangian.

Any conjugacy class $\mathcal{C} \hookrightarrow G$ is a q-Hamiltonian G-space. $\boxed{\Rightarrow}$:

0-shifted symplectic

$$\begin{tikzcd} M/\!\!/_{\mathcal{C}} G = \left[\mathbb{R}\mu^{-1}(1)/G\right] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {[\mathcal{C}/G]} \arrow[d] \\ {[M/G]} \arrow[r] & {[G/G]} \end{tikzcd}$$

Computation of $M/\!\!/_\theta G$: assume for simplicity that $M = \mathrm{Spec}(A)$ is affine (it is automatically underived and smooth because it is symplectic)

Then $\mathbb{R}\mu^{-1}(0) = \mathrm{Spec}(B)$, where

$$B^\# = A \otimes \mathrm{Sym}(\mathfrak{g}[1]) \text{ with } \delta(a\otimes 1) = 0 \text{ for } a \in A$$

$$\& \ \delta(1\otimes x) = \mu^* x \otimes 1 \text{ for } x \in \mathfrak{g}$$

obvious action of G ("action on A $\otimes$ adjoint action")

(viewed as a linear function)

There is a map from the Cartan mixed complex

$\left(\mathrm{Sym}^\bullet_B\left(\Omega^1_B[-1]\right) \otimes \mathrm{Sym}(\mathfrak{g}^*[-2])\right)^G$ with internal differential $= \delta +$ Cartan diff.

de Rham diff $= d_{dR}$ on B.

to $\mathrm{DR}\left[\mathrm{Spec}(B)/G\right]$.

The symplectic structure is $\boxed{\omega_A + \sum_i dx_i \cdot x_i^*}$ (x_i) basis of $\mathfrak{g}$

The tangent complex is the $B \rtimes G$-module

$$\underset{-1}{\mathfrak{g}\otimes B} \longrightarrow \underset{0}{T_A \otimes_A B} \xrightarrow{T_\mu} \underset{1}{\mathfrak{g}^* \otimes B}$$