2. Shifted symplectic derived schemes

"Recall": X smooth affine algebraic variety : $A = \mathcal{O}(X)$, $T_A \overset{def}{=} Der(A) = \mathfrak{X}(X)$ projective A-mod of fin. rank

($\Leftrightarrow$ $\Omega^1_A = \Omega^1(X)$ projective A-mod of fin. rank).

De Rham complex $Sym_A(\Omega^1_A[-1])$ ~~...~~ in degree n: $\Lambda^n_A(\Omega^1_A) = \Omega^n_A$.

differential: $d(a_0 da_1 \wedge \cdots \wedge da_n) = da_0 \wedge da_1 \wedge \cdots \wedge da_n$.

A symplectic structure on X is a d-closed 2-form $\omega \in \Omega^2(X) = \Omega^2_A$ such that the induced (T_A, ω) is a symplectic vector space.

(2.1) A bit of derived "geometry"

The derived/homotopy philosophy: resolve problems before they appear.

Concretely: always take resolutions before applying functors.

A cdga$^{\leq 0}_k$. $k \longrightarrow \tilde{A} \longrightarrow A$

quasi-free : $\tilde{A}^\# = Sym(V)$ with $V \in Vect^{gr, \leq 0}$; quasi-iso

[whenever A_0 is finitely generated, one may just ask $\tilde{A}^\# = Sym_{\tilde{A}_0}(P)$ with $\tilde{A}_0$ smooth and $P \in \tilde{A}_0\text{-mod}^{gr,<0}$.]

REMARK

A-mod $\simeq \tilde{A}$-mod (as dg-categories)

$A \otimes_{\tilde{A}} M \longmapsfrom M$

Example : $A = k[x]/_{x^2}$ is not smooth. $\tilde{A} = k[x, \underset{\deg -1}{\xi}] = k(x)\xi \oplus k(x)$ with $\delta(\xi) = x^2$.

Tangent and cotangent complexes : $\mathbb{T}_A \overset{def}{=} T_{\tilde{A}}$ and $\mathbb{L}_A \overset{def}{=} \Omega^1_{\tilde{A}}$

Example : $A = k(x)/_{x^2}$ $\mathbb{L}_A = k[x,\xi]\underset{\deg -1}{d\xi} \oplus k[x,\xi]dx$, $\delta(\xi) = x^2$ & $\delta(d\xi) = d(\delta\xi) = 2x dx$.

$\mathbb{T}_A = k(x,\xi]\frac{\partial}{\partial x} \oplus k[x,\xi]\underset{\deg 1}{\frac{\partial}{\partial \xi}}$, $\langle \delta(\frac{\partial}{\partial x}), d\xi \rangle = \langle \frac{\partial}{\partial x}, \delta d\xi \rangle = 2x$

I.e. $\delta(\frac{\partial}{\partial x}) = 2x \frac{\partial}{\partial \xi}$.

$\boxed{\text{Remark}}$: $\mathbb{T}_A$ is (-1)-shifted symplectic with $\omega(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \xi}) = 1$

Well-behaved intersection theory : want to compute a "fiber product" $X \overset{h}{\underset{Z}{\times}} Y$, i.e. compute a tensor product

$D = A \overset{\mathbb{L}}{\underset{C}{\otimes}} B = \tilde{A} \underset{C}{\otimes} B$ where $C \longrightarrow \tilde{A} \longrightarrow A$

quasi free, ; quasi-iso

meaning that $\tilde{A}^\# = Sym_{..}(P)$ $P \in C^\#$-mod quasi free, quasi-iso

meaning that $\tilde{A}^\# = Sym_{C^\#}(P)$ $\quad P \in C^\#\text{-mod}$

Remark: $C \to \tilde{A}$ quasi-free means that $\tilde{X} \to Z$ submersive.

Nice feature: $\mathbb{T}_D \simeq \mathrm{hofib}\left(D \underset{A}{\otimes} \mathbb{T}_A \oplus D \underset{B}{\otimes} \mathbb{T}_B \longrightarrow D \underset{C}{\otimes} \mathbb{T}_C\right)$

I.e. $\begin{array}{ccc} X & & Y \\ & f\searrow \quad \swarrow g & \\ & Z & \end{array}$ $\qquad \mathbb{T}_{X \overset{h}{\underset{Z}{\times}} Y} \simeq \mathrm{hofib}\left(\text{"}\mathbb{T}_X \oplus \mathbb{T}_Y \to \mathbb{T}_Z\text{"}\right)$.

Example: $X = Y = \mathbb{A}^1 \hookrightarrow \mathbb{A}^2$. $\quad A = B = k[x] \longleftarrow k[x,y] = C$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \longleftarrow y$

The self (derived) intersection of $\mathbb{A}^1$ in $\mathbb{A}^2$ is computed as follows:

- $C \to \tilde{A} = k[x,y,\xi] \longrightarrow A$
  $\delta\xi = y$
- $D = \tilde{A} \underset{C}{\otimes} B = k[x,\xi]$. This is $\mathbb{A}^1 \times \mathbb{A}^1[-1]$ — the "odd line"
- $\mathbb{T}_{k[x,\xi]} = k[x,\xi]\frac{\partial}{\partial x} \oplus k[x,\xi]\frac{\partial}{\partial \xi}$

$$\mathrm{hofib}\left(D \underset{\tilde{A}}{\otimes} T_{\tilde{A}} \oplus D \underset{A}{\otimes} T_B \to D \underset{A}{\otimes} T_C\right) = \begin{array}{ccc} k[x,\xi]\frac{\partial}{\partial x} & & k[x,\xi]\frac{\partial}{\partial \xi} \\ \oplus & \nearrow & \oplus \\ k[x,\xi]\frac{\partial}{\partial y} & \searrow & k[x,\xi]\frac{\partial}{\partial x} \\ \oplus & \times & \oplus \\ k[x,\xi]\frac{\partial}{\partial x} & & k[x,\xi]\frac{\partial}{\partial y} \\ \uparrow & & \uparrow \\ \text{degree } 0 & & \text{deg } 1 \end{array}$$

$D \underset{\tilde{A}}{\otimes} T_{\tilde{A}}$

$D \underset{A}{\otimes} T_B$

$D \underset{A}{\otimes} T_C$

acyclic part

acyclic part

on generators: $\ker = k\frac{\partial}{\partial x}$ & $\mathrm{coker} = k\frac{\partial}{\partial \xi}$

(2.2) De Rham complex

$DR^\bullet(A) \overset{def}{=} Sym^\bullet_{\tilde{A}}\left(\Omega^1_{\tilde{A}}[-1]\right)$. There are two commuting differentials:

1) δ (internal differential), which preserves the symmetric weight

2) d (de Rham differential), wich increases sym. weight by 1.

This structure is called a graded mixed complex (equivalent, up to grading conventions, to bicomplexes).

Definitions: • a **2-form of degree n** is an $(n+2)$-cocycle in $Sym^2_{\tilde{A}}\left(\Omega^1_{\tilde{A}}[-1]\right)$ ← differential is δ

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \simeq$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Lambda^2_{\tilde{A}}(\Omega^1_{\tilde{A}})[-2]$

A_{SI} differential is δ

$\Lambda^2_{\tilde{A}}(\Omega^1_{\tilde{A}})[-2]$.

- Such a 2-form induces a cochain map $\Lambda^2_{\tilde{A}} T_{\tilde{A}} \to A[n]$.
  It is said non-degenerate if this pairing is.
- a closed 2-form of degree n is an $(n+2)$-cocycle in $\prod_{\alpha \geq 2} \mathrm{Sym}^{\alpha}_{\tilde{A}}(\Omega^1_{\tilde{A}}[-1])$, for the total differential $\delta + d$. Explicitly:

  This is a series $\omega_0 + \omega_1 + \omega_2 + \cdots$ with ω_i of weight $2+i$ and

  $$\delta\omega_0 = 0 \quad (\text{i.e. } \omega_0 \text{ is a 2-form of degree } n)$$

  $$\forall i \geq 0, \quad d\omega_i + \delta\omega_{i+1} = 0. \quad (\text{e.g. } \omega_1 \text{ is a homotopy between } d\omega_0 \text{ and } 0)$$
- an n-shifted symplectic structure on A (or $X = \mathrm{Spec}(A)$) is a closed 2-form of degree n ω such that ω_0 is ND.

Examples:
- assume A carries an 0-shifted symplectic structure.

  Then $\omega_0^\flat : \underset{[0,1,\ldots)}{\mathbb{T}_A} \to \underset{(\ldots,-1,0]}{\mathbb{L}_A}$ is a quasi-isomorphism.

  $\Rightarrow H^0(\mathbb{T}_A) \xrightarrow{q\text{-iso}} \mathbb{T}_A \xrightarrow{q\text{-iso}} \mathbb{L}_A \xrightarrow{q\text{-iso}} H^0(\mathbb{L}_A)$

  $\Rightarrow$ A is concentrated in degree 0 and smooth, and honestly symplectic.
- assume A carries a (-1)-shifted symplectic structure.

  Then $\omega_0^\flat : \mathbb{T}_A \to \mathbb{L}_A[-1]$ is a quasi-isomorphism.

  $\Rightarrow$ $\mathbb{T}_A$ has cohomology in degrees 0 & 1, of finite rank.

  One says that A is quasi-smooth (defect of smoothness is controlled by a single vector bundle: the obstruction bundle).

Remarks:
- for obvious degree reasons, there are no n-shift symplectic structures on A for $n > 0$.
- Work of Brav–Bussi–Joyce shows that, locally, n-shifted symplectic structures on A have strict normal forms (ie. $d\omega_0 = 0$, $\omega_1 = \omega_2 = \cdots = 0$ and $\omega_0^\flat$ is an isomorphism).

(2.3) Lagrangian morphisms

Definition: Let $Y \xrightarrow{f} X$ a morphism represented by $A \xrightarrow{f^*} B$.

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Assume X (ie A) is equipped with an n-shifted symplectic structure ω.
A Lagrangian structure on f (w.r.t. ω) is a homotopy η between $f^*\omega$ and 0
such that η_0 (the induced homotopy between $f^*\omega_0$ and 0) is ND.

Concretely: $\eta = \eta_0 + \eta_1 + \eta_2 + \cdots$ with η_i of weight $i+2$ and $\omega = (d+\delta)(\eta)$.

In particular $\omega_0 = \delta \eta_0$.

Same features as in the linear setting!

(1) Lagrangian structures on $X \to *$ $\iff$ $(n-1)$-shifted symplectic structures on X.

$*$ equipped with the null n-shifted sympl. structure

(2) Lagrangian correspondences compose well.

(1) + (2) $\Rightarrow$ Lagrangian (derived) intersections in n-shifted symplectic are $(n-1)$-shifted symplectic

Example [derived critical locus] $X = \mathrm{Spec}(A)$ smooth affine alg.-variety.

T^*X is symplectic.

$X \xrightarrow{0} T^*X$ zero section is lagrangian.

$X \xrightarrow{\lambda} T^*X$ the graph of any closed 1-form λ as well.

$$\mathbb{R}Z(\lambda) = X \underset{T^*X}{{}_{\lambda}\overset{h}{\times}_{0}} X \text{ is } (-1)\text{-shifted symplectic}$$

Whenever $\lambda = df$, $\mathbb{R}Z(\lambda) = \mathbb{R}\mathrm{Crit}(f)$ is the derived critical locus of f.

- Computation for $X = \mathrm{Spec}(k[x])$: $\lambda = \alpha(x)dx$.

$$T^*X = \mathrm{Spec}(k[x,y]) \quad \omega = dx \wedge dy = \omega_0$$

$$\begin{tikzcd} & k[x,y] \arrow[r, "y=\alpha(x)"] \arrow[dl, "\delta\xi = y"'] \arrow[d, "y=0"] & k[x] \arrow[d, dashed] \\ k[x,y,\xi] & k[x] & \end{tikzcd}$$

homotopy for ω in $k[x,y,\xi]$.

$\delta\xi = 1$

$$k[x,y,\xi] \xrightarrow{\sim} k[x] \dashrightarrow k[x,\xi]$$

$y=0$

$\delta\xi = \alpha(x)$

i.e. $\delta = \iota_\lambda$

homotopy for ω in $k[x,y,\xi]$.

$\eta_0 = \eta = dx \wedge d\xi$:

$\delta\eta = \delta dx \wedge d\xi \pm dx \wedge \delta d\xi = dx \wedge dy = \omega$

& $d\eta = 0$

$\Rightarrow$ the (-1)-shifted symplectic structure on $(k[x,\xi], \iota_\alpha)$ is $dx \wedge d\xi$.

More generally $\mathbb{R}Z(\lambda)$ is given by $\left(\underbrace{\mathrm{Sym}_A(T_A[1])}_{\text{polyvector fields in } \leq 0.}, \iota_\lambda\right)$

Ex: If $\lambda = df = x^2 dx$ $\left(f = \frac{x^3}{3}\right)$ then $\mathbb{R}Z(\lambda) = \mathbb{R}\mathrm{Crit}\left(\frac{x^3}{3}\right) \simeq k[x]/x^2$