1.2 Lagrangian structures

Recall the cocone of a cochain map $(V,d_V) \xrightarrow{\varphi} (W,d_W)$: $\mathrm{hofib}(\varphi) = \left(\begin{matrix} V \\ \oplus \\ W[-1] \end{matrix} , d \right)$

aka "mapping cocone" or "homotopy fiber"

with $d = \begin{pmatrix} d_V & 0 \\ \varphi & -d_W \end{pmatrix}$

Main feature: a cochain map $(U, d_U) \longrightarrow \mathrm{hofib}(\varphi)$ coincides with the data of a cochain map $(U, d_U) \xrightarrow{\psi} (V, d_V)$ together with a homotopy $\varphi\psi \simeq 0$ [recall that is $\eta : C \longrightarrow W[-1]$ such that $\eta d_C + d_W \eta = \varphi\psi$]

Indeed: $$\begin{pmatrix} \psi d_C \\ \eta d_C \end{pmatrix} = \begin{pmatrix} \psi \\ \eta \end{pmatrix} d_C = d \begin{pmatrix} \psi \\ \eta \end{pmatrix} = \begin{pmatrix} d_V & 0 \\ \varphi & -d_W \end{pmatrix} \begin{pmatrix} \psi \\ \eta \end{pmatrix} = \begin{pmatrix} d_V \psi \\ \varphi\psi - d_W \eta \end{pmatrix}$$

Definition of an isotropic structure

Let (V, ω) be a complex together with $\omega : \Lambda^2 V \to k[n]$.

Let $L \xrightarrow{\varphi} V$ be a cochain map. An isotropic structure on φ (w.r.t. ω) is a homotopy $\omega|_L \simeq 0$.

Concretely, $\eta : \Lambda^2 L \to k[n-1]$ is such that $\eta(da \wedge b) \pm \eta(a \wedge db) = \omega(\varphi(a) \wedge \varphi(b))$ for $a, b \in L$.

$\Rightarrow$ $\eta^\flat$ provides a homotopy between $\varphi^* \omega^\flat \varphi$ and 0.

$\Rightarrow$ we have a morphism of complexes $L \xrightarrow{\begin{pmatrix} \varphi \\ \eta^\flat \end{pmatrix}} \mathrm{hofib}(\varphi^* \omega^\flat)$

Non-degeneracy condition

We say that η is non-degenerate if $\begin{pmatrix} \varphi \\ \eta^\flat \end{pmatrix}$ is a quasi-isomorphism.

This is equivalent to requiring that the long sequence $\cdots \to H^\bullet(L) \to H^\bullet(V) \to H^\bullet(L^*[n]) \to H^{\bullet+1}(L) \to \cdots$ is exact.

Remark: we could have asked that $L \xrightarrow{\begin{pmatrix} \omega^\flat \varphi \\ \eta^\flat \end{pmatrix}} \mathrm{hofib}(\varphi^*)$ is a quasi-isomorphism.

This is actually equivalent!

Consequence: we have a morphism between long exact sequences

Consequence: we have a morphism between long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^\bullet(L) \arrow[r] \arrow[d, equal] & H^\bullet(V) \arrow[r] \arrow[d] & H^\bullet(L^*[n]) \arrow[r] \arrow[d, equal] & H^{\bullet+1}(L) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & H^\bullet(L) \arrow[r] & H^\bullet(V^*[n]) \arrow[r] & H^\bullet(L^*[n]) \arrow[r] & H^{\bullet+1}(L) \arrow[r] & \cdots \end{tikzcd}$$

$\Rightarrow$ $H^\bullet(V) \to H^\bullet(V^*[n])$ is an iso, i.e. ω is ND.

A non-degenerate isotropic structure is called a Lagrangian structure.

Examples: • Y $(n+1)$-dimensional compact oriented manifold with boundary $\partial Y = X$.

$V = (\Omega^\bullet(X), d_{dR})[1]$ equipped with $\omega(\alpha \wedge \beta) \overset{\text{def}}{=} \int_X \alpha \wedge \beta$.

From previous lecture: it is $(2-n)$-shifted symplectic.

Let $L = (\Omega^\bullet(Y), d_{dR}) \xrightarrow{\varphi} (\Omega^\bullet(X), d_{dR})$; $\alpha \longmapsto \alpha_{|X}$.

Isotropic structure: $\eta(\alpha \wedge \beta) \overset{\text{def}}{=} \int_Y \alpha \wedge \beta$.

The homotopy property is given by Stokes' formula:

$$\eta(d\alpha \wedge \beta \pm \alpha \wedge d\beta) = \int_Y d\alpha \wedge \beta \pm \alpha \wedge d\beta = \int_Y d(\alpha \wedge \beta) = \int_X (\alpha \wedge \beta)_{|X} = \omega(\alpha_{|X} \wedge \beta_{|X}).$$

Non-degeneracy of η: the cohomology of $\mathrm{hofib}(\varphi^* : V^*[n] \to L^*[n])$ is dual (up to a shift) to the relative cohomology $H^\bullet(Y, X)$, that is itself dual (up to the same shift) to $H^\bullet(X) = H^{\bullet-1}(L)$.

Non-degeneracy thus follows from relative Poincaré duality!

• Same works with the following variation: $\mathfrak{g}$ Lie alg. with invariant ND pairing

(P, ∇) flat principal G-bundle on Y

$V = (\Omega^\bullet(X, ad(P_{|X})), \nabla)[1]$

$L = (\Omega^\bullet(Y, ad(P))), \nabla)[1]$.

Weinstein's "symplectic creed": everything is a lagrangian

A surprising example [Symplectic is lagrangian]

Let $V = 0$, equipped with the zero n-shifted symplectic structure $\omega = 0$.

A surprising example [symplectic is lagrangian]

Let $V = 0$, equipped with the zero n-shifted symplectic structure $\omega = 0$.

Q: What is a lagrangian structure on $L \to 0$?

A: $\eta : \Lambda^2 L \to k[n-1]$ such that $\underline{\eta(d a \wedge b) \pm \eta(a \wedge d b) = 0}$, that is $\underline{ND}$.

This tells us that η is a cochain map, i.e. is a degree $n-1$ skew-symmetric pairing on L

The map $L \xrightarrow{\binom{0}{\eta^b}} \mathrm{hofib}(0 \to L^*[n]) = L^*[n-1]$ is a quasi-isomorphism, i.e. η is an $(n-1)$-shifted symplectic structure on L !!!

Let's play (again!) with 2-term complexes

$V = \Big(\underset{\text{degree } d}{E} \xrightarrow{a} \underset{\text{degree } d+1}{F} \Big)$ with n-shifted symplectic structure ω, for $n = -1-2d$, determined by $\omega_L : E \otimes F \to k$.

$\left(\alpha \overset{\text{def}}{=} \omega_L^b : E \longrightarrow F^*\right)$

Let now $L = \Big(\underset{(d)}{E} \xrightarrow{b} \underset{(d+1)}{B} \Big)$ and $\varphi : L \to V$ given as $\varphi = \begin{pmatrix} id_E \\ f \end{pmatrix}$, $f : B \to F$.

Q: What is an isotropic structure on φ?

A: $\eta : \Lambda^2 B \to k$ such that for $x \in E$ and $y \in B$ $\eta(bx \wedge y) = \omega_L(x \wedge f(y))$.

$\Leftrightarrow \eta^b \circ b = f^* \circ \alpha \Leftrightarrow b^* \circ \eta^b = \alpha^* \circ f$.

In other words, η^b provides a homotopy between the composed map

$$\begin{array}{ccccccc} L & \xrightarrow{\varphi} & V & \xrightarrow{\omega^b} & V^*[n] & \xrightarrow{\varphi^*} & L^*[n] \\ \| & & \| & & \| & & \| \\ \begin{pmatrix} E \\ \downarrow b \\ B \end{pmatrix} & \begin{matrix} \xrightarrow{id_E} \\ \\ \xrightarrow{f} \end{matrix} & \begin{pmatrix} E \\ \downarrow a \\ F \end{pmatrix} & \begin{matrix} \xrightarrow{\alpha} \\ \\ \xrightarrow{\alpha^*} \end{matrix} & \begin{pmatrix} F^* \\ \downarrow a^* \\ E^* \end{pmatrix} & \begin{matrix} \xrightarrow{f^*} \\ \\ \xrightarrow{id_{E^*}} \end{matrix} & \begin{pmatrix} B^* \\ \downarrow b^* \\ E^* \end{pmatrix} \end{array}$$

and $0 : L \to L^*[n]$.

Hence the map $L \xrightarrow{\binom{\varphi}{\eta^b}} \mathrm{hofib}(\varphi^* \circ \omega^b)$ is:

$$\begin{array}{lccc} \text{degree } d & E & \xrightarrow{id_E} & E \\ & \downarrow b & & \downarrow a \oplus f^* \circ \alpha \\ \text{degree } d+1 & B & \xrightarrow{f \oplus \eta^b} & F \oplus B^* \\ & & & \downarrow (\alpha^*, b^*) \\ \text{degree } d+2 & & & E^* \end{array}$$

Assuming that E, F, B are finite dimensional and that $\alpha : E \to F^*$ is an isomorphism, degree $d+2$ $\qquad$ L

Assuming that E, F, B are finite dimensional, and that $\alpha : E \to F^*$ is an isomorphism, then the above is a quasi-isomorphism if and only if

$$\begin{tikzcd} E \arrow[r, "id_E"] \arrow[d, "b"] & E \arrow[d, "f^*\circ\alpha"] \\ B \arrow[r, "\eta^b"] & B^* \end{tikzcd}$$

is a quasi-isomorphism.

because the projection

$$\begin{tikzcd} E \arrow[d] \arrow[r] & E \arrow[d] \\ F\oplus B^* \arrow[r, two heads] \arrow[d] & B^* \arrow[d] \\ E^* \arrow[r] & 0 \end{tikzcd}$$

is a q-iso!

I.e. if and only if η^b is an isomorphism.

Example: Recall the 1-shifted symplectic structure on the 2-term complex $\mathcal{O}(\mathfrak{g}^*)\otimes\mathfrak{g} \longrightarrow \mathcal{X}(\mathfrak{g}^*)$

Let $\mu : X \longrightarrow \mathfrak{g}^*$ be a G-equivariant map.

$\mu^* : \mathcal{O}(\mathfrak{g}^*)\text{-mod} \longrightarrow \mathcal{O}(X)\text{-mod}$ is symmetric monoidal and exact

$\boxed{\Rightarrow}$ $V := \mathcal{O}(X)\otimes\mathfrak{g} \xrightarrow{a} \Gamma(X, \mu^* T_{\mathfrak{g}})$ has a 1-shifted symplectic structure.

$f = \mu_*$

Let $L := \mathcal{O}(X)\otimes\mathfrak{g} \xrightarrow{b} \mathcal{X}(X)$

Assume X carries a G-invariant 2-form $\eta \in \Omega^2(X)^G$

This defines a map $\Lambda^2_{\mathcal{O}(X)}\mathcal{X}(X) \longrightarrow \mathcal{O}(X)$ in $\mathcal{O}(X)\rtimes G$-modules.

Q: When does η defines an isotropic structure?

A: Whenever $\eta(\vec{v}_X \wedge -) = \mu^*\alpha(v)$ $\quad v \in \mathfrak{g} \hookrightarrow \mathcal{O}(\mathfrak{g}^*)\otimes\mathfrak{g} = \Omega^1(\mathfrak{g}^*)$

I.e. whenever μ is a moment map for η.

Q: When is this isotropic structure ND?

A: Since α is an iso in this situation, ND $\Leftrightarrow$ η^b iso

I.e. η (almost) symplectic.

[Alekseev-Malkin-Meinrenken]

With more work, one can prove that generalized moment maps (Lie group valued moment maps, and more generally moment maps with values in quasi-symplectic groupoids [Xu]) define Lagrangian structures.

1.3 Lagrangian correspondences

A Lagrangian correspondence is the data of (Y_1, ω_1), (Y_2, ω_2) two n-shifted symplectic

1.3 Lagrangian correspondences

A **Lagrangian correspondence** is the data of (V_1,ω_1), (V_2,ω_2) two n-shifted symplectic complexes and a Lagrangian morphism $L \to (V_1 \oplus V_2, \omega_1 - \omega_2)$.

They compose well. $\boxed{\text{WARNING}}$ one shall use homotopy fiber products!

$$\begin{tikzcd} & & & L_{13} \arrow[dl, dashed] \arrow[dr, dashed] & & & \\ & & L_{12} \arrow[dl, "a_1"'] \arrow[dr, "a_2"] & & L_{23} \arrow[dl, "b_2"'] \arrow[dr, "b_3"] & & \\ & V_1 & & V_2 & & V_3 & \end{tikzcd}$$

$$L_{13} \overset{\text{def}}{=} \operatorname{hofib}\left(L_{12} \oplus L_{23} \xrightarrow{a_2 - b_2} V_2\right)$$

$$= \begin{matrix} L_{12} \\ \oplus \\ L_{23} \\ \oplus \\ V_2[-1] \end{matrix} \quad \text{with differential} \quad \begin{pmatrix} d_{L_{12}} & 0 & 0 \\ 0 & d_{L_{23}} & 0 \\ a_2 & -b_2 & -d_{V_2} \end{pmatrix}$$

Q: What is the Lagrangian structure on $L_{13} \to (V_1 \oplus V_3, \omega_1 - \omega_3)$?

A: Recall the Lagrangian structure on $L_{12} \to (V_1 \oplus V_2, \omega_1 - \omega_2)$ is a homotopy $\eta_{12} : \Lambda^2 L_{12} \to k[n-1]$ between $a_1^* \omega_1$ and $a_2^* \omega_2$.

We also have a homotopy $\eta_{23} : \Lambda^2 L_{23} \to k[n-1]$ between $b_2^* \omega_2$ and $b_3^* \omega_3$.

Let $p : L_{13} \to L_{12}$ and $q : L_{13} \to L_{23}$ the two obvious projections.

Observe that $a_2 \circ p$ and $b_2 \circ q$ are homotopic: the homotopy is the projection $h : L_{13} \to V_2[-1]$.

homotopy between $p^* a_2^* \omega_2$ and $q^* b_2^* \omega_2$ induced by h.

$\Rightarrow$ $p^* \eta_{12} + \text{"}h^* \omega_2\text{"} + q^* \eta_{23}$ defines a homotopy between $p^* a_1^* \omega_1$ and $q^* b_3^* \omega_3$.

One can prove that it is ND whenever η_{12} and η_{23} are.

Example: $V_1 = V_3 = 0$, $V_2 = V$ honest symplectic vector space ($\Rightarrow$ 0-shifted symplectic).

L_{12} and L_{23} honest Lagrangian subspaces in V_2.

$$\begin{tikzcd} & & L_{13} \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & L_{12} \arrow[dl] \arrow[dr] & & L_{23} \arrow[dl] \arrow[dr] & \\ 0 & & V_2 & & 0 \end{tikzcd}$$

$$L_{13} = \Big(\underset{\text{degree } 0}{L_{12} \oplus L_{23}} \longrightarrow \underset{\text{degree } 1}{V_2} \Big), \quad (a,b) \longmapsto a-b$$

$L_{13} \to 0$ carries a Lagrangian structure

$\Leftrightarrow$ L_{13} is (-1)-shifted symplectic !!!

$\Leftrightarrow$ L_{13} is (-1)-shifted symplectic !!!

Q: what is this symplectic structure?

A: borrowing the notation from above, $\eta_{12} = 0$ and $\eta_{23} = 0$.
Therefore $\eta =$ "$h^*\omega$" recovers the last example of lecture 1.

Another example of a "Lagrangian intersection": $V_1 = V_3 = 0$, again.

$$V_2 = \begin{pmatrix} E \\ \downarrow a \\ F \end{pmatrix} \quad \text{and we assume that } \alpha : E \to F^* \text{ iso.}$$

$$L_{12} = \begin{pmatrix} E \\ \downarrow b \\ B \end{pmatrix} \quad \text{and} \quad L_{23} = \begin{pmatrix} 0 \\ \downarrow \\ F \end{pmatrix} \xrightarrow{\binom{0}{id_F}} V_2 \quad \text{is lagrangian:}$$

2-term example from above.

- obviously isotropic
- ND because

$$\begin{pmatrix} 0 \\ \downarrow \\ F \end{pmatrix} \to \mathrm{hofib}\begin{pmatrix} E & \xrightarrow{\alpha} & F^* \\ a\downarrow & & \downarrow \\ F & \to & 0 \end{pmatrix} = \begin{matrix} E \\ \downarrow (a,\alpha) \\ F \oplus F^* \end{matrix}$$

is a quasi-iso (indeed: α iso $\Rightarrow$ $\ker(a,\alpha) = 0$ & $\mathrm{coker}(a,\alpha) \simeq F$).

Therefore L_{13} is $(n-1)$-shifted symplectic.

$$\text{Recall } L_{13} = \mathrm{hofib}\left(\begin{matrix} E \\ \downarrow b \\ B \end{matrix} \oplus \begin{matrix} 0 \\ \downarrow \\ F \end{matrix} \longrightarrow \begin{matrix} E \\ \downarrow a \\ F \end{matrix} \right) = \begin{tikzcd}[column sep=small] & E \arrow[dl, "b"'] \arrow[dr, "id_E"] & \\ B \oplus \arrow[dr, "f"'] & F \arrow[d, "id_F"] & \oplus E \arrow[dl, "a"] \\ & F & \end{tikzcd} \longrightarrow \begin{tikzcd}[column sep=small] & E \arrow[dl, "b"'] \arrow[dr, "id_E"] & \\ B & & E \end{tikzcd} \longleftarrow B$$

acyclic part

quasi-iso

acyclic part

$\Rightarrow$ $B \simeq L_{13}$ is $(n-1)$-shifted symplectic.