## 1. Shifted symplectic linear algebra

$k$ = field of char. $0$.

**Recall**: a symplectic structure on a $k$-vector space $V$ is a linear map $\omega : \Lambda^2 V \to k$ that is non-degenerate.

non-degenerate $\overset{\text{def}}{=}$ $\omega^\flat : V \to V^*$, $v \mapsto \omega(v \wedge -)$ is an isomorphism.

**Main idea**: replace $k$-vector spaces with (cochain) complexes of $k$-vector spaces and isomorphisms with quasi-isomorphisms.

**Recall**: let $(V, \omega)$ be a symplectic vector space, and let $L \subset V$ be a subspace. $L$ is lagrangian if $\omega_{|\Lambda^2 L} = 0$ (i.e. $L$ is isotropic) and $L$ is maximal for this property.

Equivalent caracterisations of the maximality:

(a) $\dim(L) = \frac{1}{2}\dim(V)$

(b) $L \subset L^\circ \overset{\text{def}}{=} \{ v \in V \mid \omega(v \wedge -)_{|L} = 0 \}$ is an equality.

(c) the sequence $0 \to L \to V \simeq V^* \to L^* \to 0$ is exact.

**Main idea**: replace → $k$-vector spaces with complexes

→ $\omega_{|\Lambda^2 L} = 0$ by $\omega_{|\Lambda^2 L}$ is homotopic to $0$.

This implies is $\Leftrightarrow$ to $L \to V \tilde{\to} V^* \to L^*$ homotopic to $0$. recall this is a quasi-iso

→ condition (c) by "the induced long sequence in cohomology

$\cdots \to H^\bullet(L) \to H^\bullet(V) \tilde{\to} H^\bullet(V^*) \to H^\bullet(L^*) \to H^{\bullet+1}(L) \to \cdots$ is exact."

**Observation**: all the above makes sense within a symmetric monoidal abelian $k$-linear category.

E.g. $\mathrm{Mod}_A$ ($A$ commutative $k$-algebra), $\otimes_A$ ⇝ Complexes of those

$\mathrm{Rep}(G)$ ($G$ group), $\otimes_k$ ⇝ — " —

⚠ the symmetric monoidal $\mathbb{R}$-linear category of vector bundles is not abelian ⇝ but complexes of those is good enough for our purposes.

abelian

The most general context for what we do is the one of symmetric monoidal stable $\infty$-categories. We will avoid the language of $\infty$-categories as much as we can.

## 1.1 Linear shifted symplectic structures

$V$ = complex of $k$-vector spaces (or $A$-modules, or $G$-rep, or ...)

**Definition**: An n-shifted symplectic structure on $V$ is a cochain map $\omega : \Lambda^2 V \to k[n]$ such that $\omega^\flat : V \to V^*[n]$, $v \mapsto \omega(v \wedge -)$ is a quasi-isomorphism.

**Observations**:
- recall $V[n]^k = V^{k+n}$. I.e. if $V$ is concentrated in deg $0$ then $V[n]$ is concentrated in deg $\boxed{-n}$ !!!!
- in general, one shall replace $k$ with the monoidal unit $\mathbb{1}$ (that is $A$ for $\mathrm{Mod}_A$, the trivial character for $\mathrm{Rep}(G)$, ...).
- in general we should consider $\Lambda^2 \tilde{V} \to k[n]$ for any $\tilde{V}$ linked to $V$ by a sequence of quasi-isomorphism. But if $V$ is nice enough (e.g. made of projectives) this is equivalent.

We will ignore this issue

**Examples**:
- $X$ n-dimensional closed oriented manifold.

  $V = (\Omega^\bullet(X), d_{dR})[1]$ is $(2-n)$-shifted symplectic,

  with $\omega(\alpha \wedge \beta) := \int_X \alpha \wedge \beta$

  ⚠ formal $\wedge$ $\in \Lambda^2 V$ ; $\wedge$ of forms $\in V$

  This example already leads to an interesting observation: the ND of $\omega$ does not impose that $V$ is f.d. but perfect (it has f.d. cohomology & in finitely many degrees)

- $X$ n-dimensional closed oriented manifold. $G$ Lie group ($\mathfrak{g} = \mathrm{Lie}(G)$)

  $(P, \nabla)$ flat principal $G$-bundle. $\Gamma = \mathrm{Aut}(P, \nabla) \subset C^\infty(X, G)$

  $\mathrm{ad}(P) \overset{\mathrm{def}}{=} P \times_G \mathfrak{g}$.

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$$V = (\Omega^\bullet(X, \mathrm{ad}(P)), \nabla)[1]$$

For every $G$-invariant symmetric ND pairing $\langle , \rangle$ on $\mathfrak{g}$ we have a $(2-n)$-shifted symplectic structure on $V$ (viewed as a complex of $\Gamma$-repr.):

$$\omega(\alpha \wedge \beta) = \int_X \langle \alpha \overset{\wedge}{,} \beta \rangle$$

## The case of 1-term complexes

If $V$ is concentrated in degree $d^{*}$, then $V^*$ is concentrated in degree $-d$

Therefore the ND condition $V \xrightarrow{\sim} V^*[n]$ imposes that $d = -n-d$, i.e. $n = -2d$.

$*$: i.e. $V = W[-d]$ where $W$ is concentrated in degree $0$. There are two cases:

→ $d$ is odd: $\Lambda^2(V) = \mathrm{Sym}^2(W)[-2d] \longrightarrow k[-2d]$

I.e. the $n$-shifted symplectic structure on $V$ is a scalar product on $W$.

→ $d$ is even: $\Lambda^2(V) = \Lambda^2(W)[-2d] \longrightarrow k[-2d]$

I.e. the $n$-shifted symplectic structure on $V$ is an honest sympl. str. on $W$.

**Main example**: $G$ Lie group. $\mathfrak{g} = \mathrm{Lie}(G)$ equipped with a ND invariant scalar product.

$\Rightarrow \mathfrak{g}[1]$ is 2-shifted symplectic (viewed in complexes of $G$-representations).

## The case of 2-term complexes

Let $V = \underset{d}{E} \xrightarrow{a} \underset{d+1}{F}$ and let $n$ be odd.

explanation for this assumption:

If $V$ carries an $n$-shifted symplectic structure, and $n$ is even, then we must have:

either $\ker(a) = H^d(V) \xrightarrow{\sim} H^{-d-1}(V^*) = \ker(a^*) = 0$

$0 = \mathrm{coker}(a) = H^{d+1}(V) \xrightarrow{\sim} H^{-d}(V^*) = \mathrm{coker}(a^*)$

$\Rightarrow V \simeq \ker(a)[-d]$ and $n = -2d$

or $H^d(V) = 0 \quad H^{-d-1}(V^*)$ $\Rightarrow V \simeq \mathrm{coker}(a)[-d-1]$ and $n = -2d-2$

$H^{d+1}(V) \xrightarrow{\sim} H^{-d}(V^*) = 0$

We are back to 1-term complexes.

An $n$-shifted symplectic structure on $V$ is uniquely determined by $\omega_L : E \otimes F \to k$ satisfying the following properties:

① $\omega_L(e_1 \otimes a(e_2)) = \pm \omega_L(a(e_1) \otimes e_2)$

(cochain condition for $\omega$)

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② $n = -1-2d$

③ $\ker(a) \xrightarrow{\alpha} \ker(a^*)$ & $\mathrm{coker}(a) \xrightarrow{\alpha^*} \mathrm{coker}(a^*)$ are iso.

We are back to 1-term complexes.

this is equivalent to requiring that

$$V = \begin{pmatrix} E^{(d)} \\ \downarrow a \\ F^{(d+1)} \end{pmatrix} \xrightarrow{\omega^\flat = \binom{\alpha}{\alpha^*}} \begin{pmatrix} F^{*\,(-d-1-n)} \\ \downarrow a^* \\ E^{*\,(-d-n)} \end{pmatrix} = V^*[n]$$

is a cochain map $\left(\alpha \overset{\mathrm{def}}{=} \omega^\flat_L\right)$.

**Lemma**: Assume $E$ and $F$ are f.d. Then

③ $\iff \ker(a) \xrightarrow{\alpha} \ker(a^*)$ iso

$\iff \ker(a) \cap \ker(\alpha) = 0$ and $\dim(E) = \dim(F)$

$\iff \ker(\alpha) \xrightarrow{a} \ker(\alpha^*)$ iso

The proof is a linear algebra exercise!!!

## Examples

1) $G$ Lie group (or affine algebraic group). $\mathfrak{g} = \mathrm{Lie}(G)$. $A = \mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g})$.

$A$ is a $G$-algebra (because $\mathfrak{g}^*$ is a $G$-space): $g \in G$, $x \in \mathfrak{g}$, $g\cdot(x^\ell) \overset{\mathrm{def}}{=} (\mathrm{Ad}_g(x))^\ell$ $(\ell \in \mathbb{N})$.

Consider the infinitesimal action map
$$\mathfrak{g} \to \mathcal{X}(\mathfrak{g}^*)$$
$$x \mapsto \vec{x}$$

It induces an $A \rtimes G$-module map
$$a : A \otimes \mathfrak{g} \to \mathcal{X}(\mathfrak{g}^*) \simeq A \otimes \mathfrak{g}^*$$
$$f \otimes x \mapsto f\vec{x}$$

We view it as a 2-term complex (of $A \rtimes G$-modules) in degree $-1$ & $0$.

Let us define $\omega_L : (A \otimes \mathfrak{g}) \underset{A}{\otimes} (A \otimes \mathfrak{g}^*) = A \otimes \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} A$

It induces an ISOMORPHISM of 2-term complexes:

$$\begin{pmatrix} A \otimes \mathfrak{g} \\ \downarrow a \\ A \otimes \mathfrak{g}^* \end{pmatrix} \xrightarrow{\mathrm{id}} \begin{pmatrix} A \otimes \mathfrak{g} \\ \downarrow a^* \\ A \otimes \mathfrak{g}^* \end{pmatrix}$$

The only thing to check is that $a = a^*$: if $(e_i)_{i=1,\dots,n}$ is a basis of $\mathfrak{g}$ then $a(e_i) = c_{ij}^k e_j^* e_k$. Hence $a = a^*$:
$$\langle a^* e_i, e_j \rangle = -\langle a(e_j), e_i \rangle = -c_{ji}^k e_k$$
$$= c_{ij}^k e_k = \langle a(e_i), e_j \rangle.$$

This gives an example of a 1-shifted symplectic structure.

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$= c_{ij}^k e_k = \langle a(e_i), e_j \rangle$.

2) Now we assume $G$ is reductive (or compact in the real setting) and we choose a ND invariant pairing $\langle , \rangle : \mathrm{Sym}^2(\mathfrak{g}) \to k$ on $\mathfrak{g}$.

$A = \mathcal{O}(G)$ is a $G$-algebra (because $G$ is a $G$-space conjugation).

We consider the infinitesimal action map

$$a : \mathfrak{g} \longrightarrow \mathcal{X}(G) \simeq A \otimes \mathfrak{g}$$
$$x \longmapsto \vec{x} = x^L - x^R$$

in the left trivialization $TG \simeq G \times \mathfrak{g}$ this is the section $G \ni g \mapsto (g, x - Ad_g(x))$

Giving us a 2-term complex of $A \rtimes G$-modules:

$$\underset{(-1)}{A \otimes \mathfrak{g}} \xrightarrow{a} \underset{(0)}{\mathcal{X}(G)}$$
$$f \otimes x \longmapsto f \vec{x}$$

Let us define $\omega_L : (A \otimes \mathfrak{g}) \underset{A}{\otimes} \mathcal{X}(G) = \mathfrak{g} \otimes \mathcal{X}(G) \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{O}(G) \xrightarrow{\langle , \rangle \otimes id} \mathcal{O}(G)$

Use the average of left & right MC forms: $\frac{1}{2}(g^{-1}dg + dg \cdot g^{-1}) \in (\mathfrak{g} \otimes \Omega^1(G))^G$

Check as an exercise that the induced map

$$\alpha : A \otimes \mathfrak{g} \longrightarrow \Omega^1(G) \simeq \mathcal{X}(G)$$

is given by $\alpha(1 \otimes x) = \frac{1}{2}(x^L + x^R)$

$$\Omega^1(G) \simeq A \otimes \mathfrak{g}^* \simeq A \otimes \mathfrak{g} \simeq \mathcal{X}(G).$$

We observe that $rk_A(A \otimes \mathfrak{g}) = \dim(\mathfrak{g}) = rk_A(\Omega^1(G))$

and that $\ker(a) \cap \ker(\alpha) = \{ x \mid x^L - x^R = 0 = \frac{1}{2}(x^L + x^R) \} = 0$.

This also defines a 1-shifted symplectic structure (thanks to the Lemma).

3) The previous examples are specific cases of the following.

Let $G = \left( \begin{array}{c} G_1 \\ s \downarrow \uparrow e \downarrow t \\ G_0 \end{array} \right)$ be a Lie groupoid equipped with a multiplicative 2-form $\omega \in \Omega^2(G_1)$.

We consider the anchor map $a : \underset{(-1)}{e^* T^s G_1 = L} \xrightarrow{dt} \underset{(0)}{TG_0}$

viewed as a $G$-equivariant 2-term complex of vector bundles on $G_0$.

Lie algebroid associated with $G$

We define $\omega_L$ to be the restriction of $\omega$ to $L \times TG_0 \subset e^*(TG_1 \times TG_1)$

One can check that multiplicativity of $\omega \Rightarrow \omega_L$ satisfies the cochain condition.

$\omega$ is called an almost quasi-symplectic structure on $G$ if $a : \ker(\alpha) \to \ker(\alpha^*)$ is an iso,
[P. Xu]
i.e. if $\omega_L$ defines a 1-shifted symplectic structure.

4) Let $(V, \omega)$ a usual symplectic vector space and $L_1, L_2 \subset V$ Lagrangian subspaces (in the usual sense).

Consider the 2-term complex

$$\underset{\text{degree } 0}{L_1 \oplus L_2} \longrightarrow \underset{\text{degree } 1}{V}$$
$$(a,b) \longmapsto a-b$$

Define $\omega_L : (L_1 \oplus L_2) \otimes V \longrightarrow k$

$$(a,b) \otimes v \longmapsto \omega(a+b, v)$$

It satisfies the cochain condition: $\omega(a+b, a'-b') - \omega(a'+b', a-b) = 2\omega(a,a') - 2\omega(b,b') = 0$
(because $L_1$ and $L_2$ are isotropic)

It also satisfies the ND condition: $\dim(L_1 \oplus L_2) = \dim(V)$ (because $L_1$ and $L_2$ are Lagrangian)

& $\{(a,b) \mid a-b=0 \text{ \& } \omega(a+b,-) \equiv 0\} = 0$
we use that $\omega$ is ND.

This defines a $(-1)$-shifted symplectic structure.

(note that we've really used all assumptions to get this).